Higher Transcendental Functions

CALIFORNIA INSTITUTE OF TECHNOLOGY

BATEMAN MANUSCRIPT PROJECT

A. Erdélyi, *Editor*

W. Magnus, F. Oberhettinger, F. G. Tricomi, *Research Associates*

Higher Transcendental Functions, 3 volumes.

Tables of Integral Transforms, 2 volumes.

ERRATA

HIGHER TRANSCENDENTAL FUNCTIONS, VOL. I.

P. 5, line 10: Read $[1-z/(2n+1)]^{-1}$ instead of $[1-z/(2n+1)^{-1}]$.

P. 12, equation (32): Read $\tan^{-1}(q/p)$ instead of $\tan^{-1}(p/q)$.

P. 13, equation (40): Add $a>0$, $p>1$.

P. 24, line 6 up: Read $\operatorname{Re} s>1$ instead of $\operatorname{Re} s>0$.

P. 34, line 6 up: Formula should read

$$\gamma_n=\frac{(-1)^n}{n!}\lim_{m\to\infty}\left(\sum_{l=1}^{m}\frac{(\log l)^n}{l}-\frac{(\log m)^{n+1}}{n+1}\right)$$

P. 41, line 5: Read $e^{z(x-\frac{1}{2})}$ instead of $e^{2(x-\frac{1}{2})}$.

P. 52, equation (4): Read b^a instead of b^{a-1}.

P. 57, second equation (4): Read $r!$ instead of $m!$.

P. 61, line 6: Read from instead of form.

P. 63, line 2 up: Insert $|$ after $|\arg(-z)$.

P. 64, equation (25): Read $1+a-b$ instead of $1-a+b$.

P. 65, line 2: Insert $\}$ before the last comma.

P. 70, equation (6): Read $r+1-m$ instead of $m+1-r$ (four times).

P. 76, equation (11): On the right-hand side read $\Gamma(a+n+1)$ instead of $\Gamma(a+n)$.

P. 78, last line: Read $F(a-a',$ instead of $F(a-a,$.

P. 86, lines 7 up and 9 up: Omit $(-1)^n$.

P. 88, line 8 up: Read 2.1(24) instead of 1.5(24).

P. 93, line 16 up: Read (1) instead of (13).

P. 104, equation (43): Insert z between $(c-a)$ and $F(c+1)$.

P. 107, equation (36): Read $\Gamma(b)$ instead of $\Gamma(c)$.

P. 108, line 15 up: Read $D \neq D'$ instead of $D = D'$.

P. 108, equations (1) and (2): Insert $|$ before $\arg(1-z)|$.

P. 110, last line: Read $(1-z)^{-2}$ instead of $(1-z)^{-1}$.

P. 112, equation (17): Read $z^{1/2}$ instead of $z_{1/2}$.

P. 112, equation (29): Read $z^2(2-z)^{-2}$ instead of $z^2/(2-z)^{-2}$.

P. 113, equation (34): On the right-hand side read $a-b+1$ instead of $a-b-1$.

P. 116, line 4 up: Delete 2.1(15).

P. 126: Insert a horizontal rule midway between (20) and (21).

P. 145, equation (23): Read $\Gamma(\tfrac{3}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)$ instead of $\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)$.

P. 154, line 10: Read $\mathrm{Re}\,\mu < 1$ instead of $\mathrm{Re}\,\nu > -1$.

P. 166, line 9: Read $(\sin v)^{\mu}$ instead of $(\sin v)^{n}$.

P. 168, line 8 up: Read $\Gamma(\nu+m+1)$ instead of $\Gamma(\nu+m)$.

P. 169, line 5: Read $0 \le \theta' < \pi$ instead of the second $0 \le \theta < \pi$.

P. 169, last line of equation (3): Read $0 < \theta < \pi/2$ instead of $0 < \theta' < \pi/2$.

P. 179, equation (32): Insert $z^{-a-2\nu}$ on the right-hand side, and read z^{-2} instead of z^2.

P. 179, last line: Read w' instead of w, and insert w before $=$ sign.

P. 182, equation (1): Read $\sum_{n=0}^{\infty}$ instead of $\sum_{n=1}^{\infty}$.

P. 183, line 17: Read $\dfrac{z^n}{n!}$ instead of $\dfrac{z}{n!}$.

P. 185, equation (1): Read $2a+2b$ instead of $a+2b$.

P. 189, equation (5): Insert ; after c on the top line of the left-hand side.

P. 189, equation (7): On the right-hand side, $b = 1-a$.

P. 192, line 3: Insert [after ${}_3F_2$.

P. 193, line 3: Read $n+1;\ 1,\ 1;$ instead of $n+1,\ 1;\ 1;$.

P. 196, equation (7): Read $\sum_{n=1}^{\infty}$ instead of $\sum_{n=0}^{\infty}$.

P. 196, equation (10): Read $1;\ k^2$ instead of $1,\ k^2$, and read $1;\ 1-k^2$ instead of $1,\ 1-k^2$.

P. 196, last line: Insert Jackson's theorem at the end of the line.

P. 197, equation (11): Read q^{-N} instead of q^{-n}.

P. 197, equation (12): On the right-hand side read q^n instead of g^n.

P. 197, after equation (12) add: provided the series on the right terminates and the series on the left converges.

P. 199, line 3: Read 503-516 instead of 495-516.

P. 206, equation (14), right-hand side: Read $E(\dots\sigma : x)$ instead of $E(\dots o : x)$.

P. 209, equation (10), right-hand side: Read $2^{2p-2q}x^2$ instead of $2^{2p-2q}x$.

P. 212, line 7: Read $\min \operatorname{Re} b_h$ instead of $\max \operatorname{Re} b_h$.

P. 215, equation (1): Read $G^{1,p}_{p,q+1}(-x|$ instead of $G^{1,p}_{p,q+1}(x|$.

P. 216, equation (4): Read $(2x^{1/2})$ instead of $(2x^{1/2)}$.

P. 216, equation (9): Read $G^{10}_{04}(x|\ a,\ b,\ 2b-a,\ b+\tfrac{1}{2})$ instead of $G^{10}_{04}(x|a,\ b,\ 2b-a,\ 2b-a+\tfrac{1}{2})$.

P. 216, equation (10): Read $\tfrac{1}{2}\pi^{-1/2}[\sin(a-b)\pi]^{-1}$ instead of $\pi^{-1/2}[\sin(a-b)\pi]^{-1}\ 2^{-5/2}$.

P. 218, equation (28): Read $[I_{b-a}(2x^{1/2})-\mathbf{L}_{a-b}(2x^{1/2})]$ instead of $[I_{b-a}(2x^{1/2})\,\mathbf{L}_{a-b}(2x^{1/2})]$.

P. 218, equation (30): Read $a+b,\ a-b,\ a$ instead of $a+b,\ a-b,\ 0$.

P. 218, equation (34): Read $G^{12}_{22}\left(x\,\middle|\,\begin{matrix}-c_1,\ -c_2\\ a-1,\ -b\end{matrix}\right)$ instead of $G^{12}_{22}\left(x\,\middle|\,\begin{matrix}a-1,\ -b\\ -c_1,\ -c_2\end{matrix}\right)$

P. 220, equation (59): Read $\frac{x^4}{64}$ instead of $\begin{matrix} x^4 \\ 64 \end{matrix}$.

P. 221, equation (65): Read $\frac{1}{2}\mu+\nu,\ \frac{1}{2}\mu-\nu,\ \frac{1}{2}\mu$ instead of $\nu+\frac{1}{2}\mu,\ -\nu+\frac{1}{2}\mu,\ 0$.

P. 221, equation (69): Erase fraction rule in G_{12}^{21}.

P. 222, equation (74): Read

$$_2F_1\,(a,\ b;\ c;\ -x)=\frac{\Gamma(c)\,x}{\Gamma(a)\,\Gamma(b)}\ G_{22}^{12}\left(x\,\middle|\,\begin{matrix}-a,\ -b\\ -1,\ -c\end{matrix}\right)$$

P. 222, equation (75), second line: Read

$$\times\, G_{44}^{14}\left(z\,\middle|\,\begin{matrix}-a,\ -b,\ -c,\ -d\\ -1,\ -e,\ -f,\ -l\end{matrix}\right)$$

P. 258, equation (10): Insert $-$ after first $=$ sign.

P. 262, equation (5): Read $[\log x+\psi(a)-2\gamma]$ instead of $\log x$.

P. 263, equation (14): second form of the result: Read $\Phi(c-a,\ c;\ \xi)$ and $\Phi(1-a,\ 2-c;\ \xi)$ instead of $\Phi(c-a,\ c;\ -\xi)$ and $\Phi(1-a,\ 2-c;\ -\xi)$.

P. 268, equation (36): Read Φ instead of ϕ.

P. 269, line 3 up: Read ; instead of : on the right-hand side.

P. 270, equation (7), second line: Read $1-s$ instead of $1-s^{-1}$.

P. 271, equation (16): Insert the factor t^{c-1} on the right-hand side.

P. 279, line 9 up: Read $x^{\frac{1}{2}c-\frac{1}{4}}$ instead of $x^{\frac{1}{4}+\frac{1}{2}c}$.

P. 294, lines 18 to 20: Read MacRobert, T.M. instead of MacRobert, R.M.

P. 295, line 4: Read Sharma, J.L. instead of Sharma, G.L.

HIGHER TRANSCENDENTAL FUNCTIONS

Volume I

Based, in part, on notes left by

Harry Bateman

Late Professor of Mathematics, Theoretical Physics, and Aeronautics at the California Institute of Technology

and compiled by the

Staff of the Bateman Manuscript Project

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This work is dedicated to the
memory of

HARRY BATEMAN

as a tribute to the imagination which led him to undertake a project of this magnitude, and the scholarly dedication which inspired him to carry it so far toward completion.

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PREFACE

The late Professor Harry Bateman of the California Institute of Technology was one of those rare scientists who, responding to the interplay between mathematical analysis and physical understanding, made outstanding contributions to American applied mathematics. His contributions to aero- and fluid mechanics, to electro-magnetic theory, to thermodynamics, to geophysics, and to a host of other fields in which his adroit mathematical skills were applied, resulted in significant advances in these fields. During his last years he had embarked upon a project whose successful completion, he believed, would prove of great value to scientists in all fields. He planned an extensive compilation of "special functions" -- solutions of a wide class of mathematically and physically relevant functional equations. He intended to investigate and to tabulate properties of such functions, inter-relations between such functions, their representations in various forms, their macro- and microscopic behavior, and to construct tables of important definite integrals involving such functions.

It is true that much of this material was already in existence. However, anyone who has been faced with the task of handling and discussing and understanding in detail the solution to an applied problem which is described by a differential equation is painfully familiar with the disproportionately large amount of scattered research on special functions one must wade through in the hope of extracting the desired information. Professor Bateman was eminently qualified to embark on such a compilation, for he was unusually familiar -- and systematically so -- with existing mathematical literature on the subject; he was exceptionally adept in mathematical analysis; and he was ever conscious of the needs of the scientist who must so often use these functions. When his death cut short his work, the California Institute of Technology, in recognition of one of its great scientists, and the Office of Naval Research, in recognition of the extremely useful service such a compilation could render to both basic and applied science, pooled their efforts to continue the task initiated by Professor Bateman.

In 1948 arrangements were completed between the California Institute of Technology and the Office of Naval Research to employ at the California Institute of Technology four mathematical analysts of international reputation to complete Professor Bateman's work: Professors Arthur Erdélyi of the University of Edinburgh; Wilhelm Magnus of the University of Göttingen; Fritz Oberhettinger of the University of Mainz; and Francesco Tricomi of the University of Torino. It was not long after this team began work that it became apparent that not only would Professor Bateman's original project find its completion in their unusually competent hands, but that the activities of such a group would lead to significant mathematical investigations and advances in the general field of mathematical analysis, as well as in the more particular field of special functions. The present compendia bear undeniable witness to the success of the undertaking.

The Office of Naval Research is proud of its collaboration with the California Institute of Technology, not only for erecting this lasting memorial to Professor Bateman, but also for producing what it considers a significant contribution to general science. These compendia, which have taken their roots in Professor Bateman's "shoe boxes" (his repository for card files) have been nurtured into mathematical maturity under the deft minds and penetrating work of the members of the international team of Erdélyi, Magnus, Oberhettinger, and Tricomi. In addition, we are pleased to have been able to render support to several young American mathematicians who have not only contributed to these compendia but were able to avail themselves of the opportunity to work and study under the direction of distinguished scientists in a field that is sorely in need of young recruits. We feel that special thanks should be extended to both Dean E. C. Watson of the California Institute of Technology and to Professor Erdélyi; to the former, for his extremely helpful and untiring interest in seeing to the establishment and completion of this task; to the latter, for assuming, in addition to scientific participation, both the scientific administrative duties of the project and the general editorial responsibilites for the publication of these compendia.

MINA REES,
Director Mathematical Sciences Division
Office of Naval Research

FOREWORD

The late Professor Harry Bateman was one of the greatest authorities in that part of mathematics now usually described as classical analysis. His knowledge of the literature was encyclopedic and probably unsurpassed and his ability to utilize this knowledge for specific problems was extraordinary. Research workers in difficulties would often write to him and receive, by return mail, detailed answers to their questions together with a list of references which in many cases amounted to a complete bibliography.

It was natural for Bateman to want to make accessible in a systematic form the tremendous amount of material which he had collected in the course of the years. His book on *Partial Differential Equations* (1932) was an attempt to carry out this task in a restricted field. Although the book was received with enthusiasm, and, after twenty years, is still one of the most important books on its subject, Bateman was not satisfied with this method of providing information. For a number of years he made plan after plan to organize and prepare for publication his material, a task made extremely difficult by the very breadth of the field which he intended to cover.

At the time of Bateman's death (1946) his notes amounted to a veritable mountain of paper. His card-catalogue alone filled several dozen cardboard boxes (the famous "shoe-boxes"). His family, his friends, and his colleagues at the California Institute of Technology very naturally wished to have some of this material prepared for posthumous publication, thereby erecting a monument to one of the most distinguished and most versatile members of the faculty of the Institute. Professor A. D. Michal, for many years a friend and colleague of the deceased, undertook the sifting of Bateman's notes. He spent several months in this herculean task, sorted out those notes which might be considered for publication and made recommendations for proceeding further with the matter. Dr. A. Erdélyi, then of the University of Edinburgh in Scotland, was invited to prepare a detailed report and proposals, and spent the academic year 1947-48 in Pasadena for this purpose.

It turned out that Bateman's notes ranged over a wider field than even his friends had suspected and also that no single section of this wide field was in a state sufficiently advanced for immediate publication. Indeed the field was so wide that it appeared imperative to narrow it down if anything useful was to be accomplished. Notes for books on functional equations, integrals in potential theory, binomial coefficients and factorials, and many other matters had to be laid aside entirely. Of the remaining material the most important part was a projected trilogy on the higher transcendental functions, on definite integrals (especially those containing higher functions), and on numerical tables offunctions occurring in applied mathematics. Since the appearance of the *Index of Mathematical Tables* by Fletcher, Miller, and Rosenhead, adequate information has been available on numerical tables, and so it was decided to concentrate on the first two parts, and these came to be called the handbook and the integral tables.

The Office of Naval Research recognized the great importance of such a work by giving generous financial support to it. Thus originated what at the California Institute came to be called the Bateman Manuscript Project. The Institute was fortunate indeed, not only in being able to persuade Professor Erdélyi to remain as its Director and as Editor of the forthcoming publications, but also in securing the services of Professor Wilhelm Magnus of the University of Göttingen (now of New York University), of Dr. Fritz Oberhettinger of the University of Mainz (now Professor at the American University, Washington, D.C.) and of Professor Francesco G. Tricomi of the University of Turin. These distinguished and internationally known scholars were assisted by a staff of younger mathematicians. The technical preparation of the vari-typescript suitable for reproduction by a photo-offset process was in the capable hands of Miss Rosemarie Stampfel.

The present volume is the first of three projected volumes on the higher transcendental functions. These three volumes will be followed by two volumes of integral tables.

The California Institute of Technology wishes to express its thanks both to the family of the late Professor Bateman for the gift of his notes and of his library, and to the Office of Naval Research and especially to Dr. Mina Rees, the Director of its Mathematical Sciences Division, for the generous support they have given to this work and for the understanding they have constantly shown for the difficulties encountered. The Institute also wishes to record its appreciation and thanks to the following persons and organizations: to Professor Michal for his preliminary survey of Bateman's notes; to the University of Edinburgh for granting leave of absence to Dr. Erdélyi; to the Rockefeller Foundation for defraying travelling expenses for Dr. and Mrs. Erdélyi on their visit in 1947-48; to the University of Turin for granting leave of absence to Professor Tricomi; to Professors T.M. Apostol of the California Institute, R. C. Archibald of Brown University, E. D. Rainville of the University of Michigan, Mr. S. O. Rice of the Bell Telephone Laboratories, and Professor C. A. Truesdell of Indiana University for information or consultations in connection with the work; and to the McGraw-Hill Company for technical advice and publication. Last but not least, acknowledgments should be expressed to Dr. Erdélyi and the staff of the Bateman Manuscript Project for the faithful and highly competent performance of a difficult task.

E. C. WATSON

Dean of the Faculty
California Institute of Technology

INTRODUCTION

The work of which this book is the first volume might be described as an up-to-date version of *Part II. The Transcendental Functions* of Whittaker and Watson's celebrated "Modern Analysis". Bateman (who was a pupil of E. T. Whittaker) planned his "Guide to the Functions" on a gigantic scale. In addition to a detailed account of the properties of the most important functions, the work was to include the historic origin and definition of, the basic formulas relating to, and a bibliography for *all* special functions ever invented or investigated. These functions were to be catalogued and classified under twelve different headings according to their definition by power series, generating functions, infinite products, repeated differentiations, indefinite integrals, definite integrals, differential equations, difference equations, functional equations, trigonometric series, series of orthogonal functions, or integral equations. Tables of definite integrals representing each function and numerical tables of a few new functions were to form part of the "Guide". An extensive table of definite integrals and a Guide to numerical tables of special functions were planned as companion works.

The great importance of such a work hardly needs emphasis. Bateman's unparalleled knowledge of the mathematical literature, past and present, and his equally exceptional diligence, would have made the book an authoritative account of its vast subject, and in many respects a definitive account; a Greater Oxford Dictionary of special functions.

A realistic appraisal of our abilities and of the time at our disposal led to a drastic revision of Bateman's plans. Only Bateman himself had the erudition to give a reliable and accurate history of special functions, and the manpower available to us was insufficient for the inclusion of all functions. Thus we restricted ourselves to an account (probably far less detailed than that planned by Bateman) of the principal properties of those special functions which we considered the most important ones. The loss thus caused to mathematical scholarship is great, regrettable, and final but we venture to hope that it will be counterbalanced in some measure by the considerable reduction in size of the book, and by the gain in the clarity of its organization. We can only hope that although

NOTATION, REFERENCES

The notation presents peculiar difficulties. There are special functions, for instance Bessel functions of the first kind, for which there is a generally accepted standard notation. There are others, like confluent hypergeometric functions, for which there are several essentially different and independent notations. The most awkward problems present themselves with those functions for which more or less the same symbol is used with several different meanings. Hermite polynomials are usually denoted by $H_n(x)$ or $He_n(x)$, but this symbol sometimes refers to the polynomials derived by repeated differentiation of $\exp(-x^2)$, and sometimes to those derived from $\exp(-\frac{1}{2}x^2)$. Moreover, some authors include, others exclude a factor, $n!$. We attempted to use the same notations throughout our book. The most significant deviation from this principle is in the case of the confluent hypergeometric series for which the symbol ${}_1F_1$ is used mostly, except in Chapter VI (and some of the later chapters) where the symbol Φ is used for the same series (and Ψ for a second solution of the confluent hypergeometric equation).

Wherever possible we followed standard notations. In the case of Bessel functions we adopted the notations used by G. N. Watson in his monumental work, in the case of orthogonal polynomials we used Szegö's notation (with the exception of using C_n^ν for ultraspherical polynomials). With Legendre functions, we followed Jahnke-Emde, Magnus-Oberhettinger and some other authors in making a distinction between the definition of Legendre functions appropriate for the interval $(-1, 1)$ and the definition appropriate for the complex plane outside of this interval. In cases of doubt we usually decided upon that notation for which more convenient or more extensive numerical tables were available. We adhered to definitions used in numerical tables even in cases in which we thought that a different definition would be preferable from the mathematical point of view. All notations are explained where they occur for the first time. There is at the end of this volume an *Index of notations* which will help the reader to find the meaning of any notation used in the book, and a *Subject index* which gives the notation for any function which occurs in the text.

Many of our chapters may be read independently of the others, yet there are many cross references. Equations within the same section are referred to simply by number, equations in other sections are indicated by the section number followed by the number of the equation. Thus (3) means equation (3) in the section in which the reference occurs, 2.1(3) means equation (3) in section 2.1. References to the literature have the name of the author followed by the year of publication. They invariably refer to the list of references at the end of the chapter.

The size and complexity of our compilation make it vain to hope that errors of judgment, or mistakes have been avoided. The undersigned will be glad to receive corrections or suggestions for the improvement of the work should a second edition become desirable.

In conclusion I should like to express the thanks of the entire project staff to the California Institute of Technology, and especially to Dean E. C. Watson, for initiating this work and for the great understanding they have shown for the numerous problems we encountered. I should also like to thank my colleagues without whose assistance the present work could not have been carried out.

A. ERDELYI

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CHAPTER I

THE GAMMA FUNCTION

1.1. Definition of the gamma function

The function $\Gamma(z)$ can be defined by one of the following expressions:

(1) $$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,dt = \int_0^1 (\log 1/t)^{z-1}\,dt \qquad \operatorname{Re} z > 0,$$

(2) $$\Gamma(z) = \lim_{n\to\infty} \frac{n!\,n^z}{z(z+1)\cdots(z+n)} = \lim_{n\to\infty} \frac{n^z}{z(1+z)(1+\tfrac{1}{2}z)\cdots(1+z/n)}$$

$$= z^{-1} \prod_{n=1}^{\infty} [(1+1/n)^z (1+z/n)^{-1}],$$

(3) $$1/\Gamma(z) = z e^{\gamma z} \prod_{n=1}^{\infty} [(1+z/n)\, e^{-z/n}],$$

where

(4) $$\gamma = \lim_{m\to\infty} \left(\sum_{n=1}^{m} 1/n - \log m\right) = 0.5772156649\cdots$$

denotes Euler's or Mascheroni's constant. The definition (1) was used by Euler, (2) (in a slightly different notation) by Gauss, and (3) by Weierstrass.

Replacing t by st in (1) (s real and positive) we get

(5) $$\Gamma(z) = s^z \int_0^\infty e^{-st} t^{z-1}\,dt \qquad \operatorname{Re} z > 0.$$

It can be shown [cf. 1.5(34)] that this formula holds for complex values of s and for a path of integration along the straight line from the origin to $\infty e^{i\delta}$. Thus we have

(6) $$\Gamma(z) = s^z \int_0^{\infty e^{i\delta}} e^{-st} t^{z-1}\,dt$$

$$-(\tfrac{1}{2}\pi + \delta) < \arg s < \tfrac{1}{2}\pi - \delta, \qquad \operatorname{Re} z > 0.$$

This equation holds for $\arg s + \delta = \pm\tfrac{1}{2}\pi$ provided $0 < \operatorname{Re} z < 1$.

From (2) and (3) it is seen that the gamma function is an analytic function of z whose only finite singularities are $z = 0, -1, -2, \ldots$. From (1) it follows that

$$(7)\quad \Gamma(z) = \int_0^1 e^{-t} t^{z-1}\, dt + \int_1^\infty e^{-t} t^{z-1}\, dt = P(z) + Q(z),$$

$Q(z)$ being an integral function. Expanding e^{-t} in a power series and integrating term by term:

$$(8)\quad P(z) = \sum_{n=0}^{\infty} (-1)^n [n!\,(z+n)]^{-1}.$$

Hence it follows that $(-1)^n/n!$ is the residue of $\Gamma(z)$ at the simple pole $z = -n$, $(n = 0, 1, 2, \ldots)$ [cf. 1.17(11)].

It will be shown that the expressions (1), (2), (3) represent the same function.

For a positive integer n and Re $z > 0$ repeated integration by parts yields

$$\int_0^n (1 - t/n)^n t^{z-1}\, dt = \frac{n!\, n^z}{z(z+1)(z+2)\cdots(z+n)},$$

so that by Tannery's theorem

$$\lim_{n\to\infty} \int_0^n (1-t/n)^n t^{z-1}\, dt = \int_0^\infty e^{-t} t^{z-1}\, dt = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)}.$$

Thus (1) is equivalent to (2). Equation (3) can be deduced from (2) as follows. By (2) we have

$$1/\Gamma(z) = \lim_{n\to\infty} z(1+z)(1+\tfrac{1}{2}z)\cdots(1+z/n)\, e^{-z\log n}$$

or

$$1/\Gamma(z) = \lim_{n\to\infty}[z(1+z)\, e^{-z}(1+\tfrac{1}{2}z)\, e^{-\frac{1}{2}z}\cdots(1+z/n)\, e^{-z/n}$$
$$\times\, e^{z(1+\frac{1}{2}+\cdots+1/n-\log n)}],$$

and finally

$$1/\Gamma(z) = z e^{\gamma z} \prod_{n=1}^{\infty} [(1+z/n)\, e^{-z/n}].$$

If the real part of z is negative, and $n + 1 > \mathrm{Re}(-z) > n$, $(n = 0, 1, 2, \ldots)$, $\Gamma(z)$ can be represented by an integral due to Cauchy and Saalschütz (Whittaker-Watson, 1927, p. 243):

$$(9)\quad \Gamma(z) = \int_0^\infty [e^{-t} - \sum_{m=0}^{n} (-t)^m/m!]\, t^{z-1}\, dt \qquad -(n+1) < \mathrm{Re}\, z < -n.$$

1.2. Functional equations satisfied by $\Gamma(z)$.

Integrating 1.1(1) by parts,

$$\Gamma(z) = (1/z) \int_0^\infty e^{-t} t^z \, dt = (1/z)\, \Gamma(1+z),$$

or

(1) $\Gamma(1+z) = z\,\Gamma(z)$,

and hence if n is a positive integer,

(2) $\Gamma(z+n) = z(z+1)(z+2) \cdots (z+n-1)\,\Gamma(z)$,

whence follows

(3) $\Gamma(z)/\Gamma(z-n) = (z-1)(z-2) \cdots (z-n)$
$= (-1)^n\,\Gamma(-z+n+1)/\Gamma(-z+1)$,

(4) $\Gamma(-z+n)/\Gamma(-z) = (-1)^n z(z-1) \cdots (z-n+1)$
$= (-1)^n\,\Gamma(z+1)/\Gamma(z-n+1)$.

Since

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1,$$

we have

$$\Gamma(n+1) = 1 \cdot 2 \cdot 3 \cdots n = n!.$$

From the expression 1.1(3),

$$\Gamma(z)\,\Gamma(-z) = -z^{-2} \prod_{n=1}^{\infty} (1 - z^2/n^2)^{-1}$$

and since

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} (1 - z^2/n^2)$$

(Bromwich, 1947, p. 294), we have

(5) $\Gamma(z)\,\Gamma(-z) = -\pi z^{-1} \csc(\pi z)$

so that

(6) $\Gamma(z)\,\Gamma(1-z) = \pi \csc(\pi z)$,

or

(7) $\Gamma(\tfrac{1}{2}+z)\,\Gamma(\tfrac{1}{2}-z) = \pi \sec(\pi z)$.

From (5) and (2)

$$(8)\quad \frac{\Gamma(n+z)\,\Gamma(n-z)}{[(n-1)!]^2} = \frac{\pi z}{\sin(\pi z)} \prod_{m=1}^{n-1} (1 - z^2/m^2) \qquad n = 1, 2, 3, \ldots .$$

From (7), (2), and (3)

$$(9)\quad \frac{\Gamma(n+\tfrac{1}{2}+z)\,\Gamma(n+\tfrac{1}{2}-z)}{[\Gamma(n+\tfrac{1}{2})]^2} = \frac{1}{\cos(\pi z)} \prod_{m=1}^{n} \left[1 - \frac{4z^2}{(2m-1)^2}\right]$$
$$n = 1, 2, 3, \ldots .$$

From (6) and (1) with $z = \tfrac{1}{2}$, it follows that

$$(10)\quad \Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-v^2}\,dv = \sqrt{\pi}.$$

We shall next prove the *multiplication formula* of *Gauss* and *Legendre*:

$$(11)\quad \prod_{r=0}^{m-1} \Gamma(z + r/m) = (2\pi)^{\frac{1}{2}(m-1)}\, m^{\frac{1}{2}-mz}\,\Gamma(mz) \qquad m = 2, 3, 4, \ldots .$$

From 1.1(2),

$$(12)\quad H(z) = \prod_{r=0}^{m-1} \Gamma(z + r/m) = \lim_{n\to\infty} n^{mz+\frac{1}{2}(m-1)}\,(n!)^m\, N^{-1},$$

where

$$N = mz(mz+1)\cdots(mz+mn)(mz+mn+1)\cdots(mz+mn+m-1)\,m^{-m(n+1)}.$$

Since

$$\Gamma(mz) = \lim_{n\to\infty} (mn)^{mz}\,(mn)!\,[mz(mz+1)\cdots(mz+mn)]^{-1},$$

we have

$$(13)\quad m^{-mz}\,\Gamma(mz)/H(z) = \lim_{n\to\infty} n^{\frac{1}{2}(m+1)}\,(mn-1)!\,(n!)^{-m}\,m^{-mn} = 1/K, \text{ say.}$$

It is evident that K is independent of z and can be evaluated by putting, for instance, $z = 1/m$ in (13). Thus

$$\Gamma(1)\,K/m = H(1/m) = \Gamma(1/m)\,\Gamma(2/m)\cdots\Gamma[(m-1)/m]\,\Gamma(1)$$

or

$$K/m = \Gamma(1 - 1/m)\,\Gamma(1 - 2/m)\cdots\Gamma[1 - (m-1)/m].$$

Multiplying the last two equations and using (6),

$$m^2\,\pi^{m-1} = K^2 \prod_{r=1}^{m-1} \sin(\pi r/m)$$

so that

(14) $K^2 = m(2\pi)^{m-1}$.

Since K, as defined by (13), is certainly positive, (12), (13), and (14) prove (11).

The case $m = 2$ of (11) is Legendre's duplication formula

(15) $\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{2})$.

1.3. Expressions for some infinite products in terms of the gamma function

From 1.1(2)

$$\frac{2\Gamma(\frac{1}{2})}{z\Gamma(\frac{1}{2}z)\Gamma(\frac{1}{2}-\frac{1}{2}z)}$$

$$= \frac{(1-z)\prod_{n=1}^{\infty}\{[1+1/n]^{\frac{1}{2}}[1+\frac{1}{2}\,1/n]^{-1}\}}{\prod_{n=1}^{\infty}[1+1/n]^{\frac{1}{2}}[1+\frac{1}{2}\,1/n]^{-1}[1+\frac{1}{2}\,z/n]^{-1}[1-z/(2n+1)]^{-1}}$$

and since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

(1) $$\frac{2\sqrt{\pi}}{z\Gamma(\frac{1}{2}z)\Gamma(\frac{1}{2}-\frac{1}{2}z)}$$

$$= (1-z)\prod_{n=1}^{\infty}[1+\tfrac{1}{2}\,z/n][1-z/(2n+1)] = (1-z)(1+z/2)(1-z/3)\cdots.$$

From 1.1(3)

(2) $$\Gamma(u)/\Gamma(u+v) = (1+v/u)\,e^{\gamma v}\prod_{n=1}^{\infty}[1+v/(u+n)]e^{-v/n}$$

$$= e^{\gamma v}\prod_{n=0}^{\infty}[1+v/(u+n)]\,e^{-v/(n+1)},$$

and hence

(3) $$\Gamma(x+iy)/\Gamma(x) = e^{-i\gamma y}\,x(x+iy)^{-1}\prod_{n=1}^{\infty}\frac{e^{iy/n}}{1+iy/(n+x)}.$$

From (2),

(4) $$\frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_3)\Gamma(z_2-z_3)} = \prod_{n=0}^{\infty}[1+z_3/(z_1+n)][1-z_3/(z_2+n)].$$

at the poles of $\Gamma(z+a)$ and $\Gamma(z+b)$. The residue of $f(z)$ at the pole $z = -(a+m)$ $(m = 0, 1, 2,)$ is

$$-\pi(-1)^m\ (m!)^{-1}\ \operatorname{ctn}(\pi a)\ \cdot \frac{\Gamma(b-a-m)}{\Gamma(c-a-m)\,\Gamma(d-a-m)},$$

and the sum of the residues at the poles of $\Gamma(z+a)$ is

$$-\pi\ \operatorname{ctn}(\pi a)\ \cdot \frac{\Gamma(b-a)}{\Gamma(c-a)\,\Gamma(d-a)}\ {}_2F_1(a-c+1, a-d+1; a-b+1; 1)$$

$$= \frac{\pi^2\ \operatorname{ctn}(\pi a)}{\sin[\pi(a-b)]} \cdot \frac{\Gamma(c+d-a-b-1)}{\Gamma(c-a)\,\Gamma(d-a)\,\Gamma(c-b)\,\Gamma(d-b)}$$

by Gauss' formula 2.1(14).

For the sum of the residues at the poles of $\Gamma(b+z)$ we have only to interchange a and b, and the sum of these two expressions is (1).

The formula

$$(2)\quad \sum_{n=0}^{\infty} (-1)^n \binom{y-1}{n} (x+n)^{-1} = \Gamma(x)\,\Gamma(y)/\Gamma(x+y) = \mathrm{B}(x, y)$$

can easily be verified by expanding the integrand of 1.5(1) in a power series and integrating term by term.

Furthermore, we have the following formulas (formulas to be proved in Chapter 2.4):

$$(3)\quad \sum_{n=0}^{\infty} \{(a)_n\,(b)_n/[(1-b+a)_n\,n!]\}\,[(\tfrac{1}{2}a+n-z)^{-1} + (\tfrac{1}{2}a+n+z)^{-1}]$$

$$= \frac{\Gamma(\tfrac{1}{2}a-z)\,\Gamma(\tfrac{1}{2}a+z)}{\Gamma(1-b+\tfrac{1}{2}a-z)\,\Gamma(1-b+\tfrac{1}{2}a+z)} \cdot \frac{\Gamma(1-b+a)\,\Gamma(1-b)}{\Gamma(a)}$$

$\operatorname{Re} b < 1,$

$$(4)\quad \frac{\Gamma(\tfrac{1}{2}a+z)\,\Gamma(\tfrac{1}{2}a-z)}{\Gamma(c-\tfrac{1}{2}a+z)\,\Gamma(1-b+\tfrac{1}{2}a-z)}$$

$$= \frac{\Gamma(a)}{\Gamma(c)\,\Gamma(1-b)} \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\,n!} \cdot (\tfrac{1}{2}a+n-z)^{-1}$$

$$+ \frac{\Gamma(a)}{\Gamma(1+a-b)\,\Gamma(c-a)} \sum_{n=0}^{\infty} \frac{(a)_n\,(1-c+a)_n}{(1-b+a)_n\,n!} \cdot (\tfrac{1}{2}a+n+z)^{-1}$$

$\operatorname{Re}(a+b-c) < 1.$

1.5. The beta function

The beta function is defined by the integral

(1) $B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1}\, dt \qquad \text{Re}\, x > 0, \quad \text{Re}\, y > 0.$

Substituting $t = v/(1+v)$, the relation

(2) $B(x, y) = \int_0^\infty v^{x-1} (1+v)^{-x-y}\, dv \qquad \text{Re}\, x > 0, \quad \text{Re}\, y > 0$

is obtained, and from this (break into $\int_0^1 + \int_1^\infty$ + let $v = \frac{1}{v}$ in latter)

(3) $B(x, y) = \int_0^1 (v^{x-1} + v^{y-1})(1+v)^{-x-y}\, dv \qquad \text{Re}\, x > 0, \quad \text{Re}\, y > 0.$

can be deduced. It follows that

(4) $B(x, y) = B(y, x).$

If

$$\int_0^\infty e^{-(1+v)t}\, t^{x+y-1}\, dt = \frac{\Gamma(x+y)}{(1+v)^{x+y}}$$

[cf. 1.1(4)] is multiplied by v^{x-1}, integrated with respect to v between 0 and ∞, and if the order of integration is inverted, we have

$$\int_0^\infty dt \int_0^\infty e^{-t(v+1)}\, t^{x+y-1}\, v^{x-1}\, dv = \Gamma(x+y) \int_0^\infty v^{x-1} (1+v)^{-x-y}\, dv$$

or

(5) $$B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)},$$

the expression for the beta function in terms of the gamma function.

The following functional equations for the beta function can be deduced easily from (4) and (5). (See Section 1.2):

(6) $B(x, y+1) = (y/x)\, B(x+1, y) = [y/(x+y)]\, B(x, y),$

(7) $B(x, y)\, B(x+y, z) = B(y, z)\, B(y+z, x) = B(z, x)\, B(x+z, y),$

(8) $$B(x, y)\, B(x+y, z)\, B(x+y+z, u) = \frac{\Gamma(x)\,\Gamma(y)\,\Gamma(z)\,\Gamma(u)}{\Gamma(x+y+z+u)},$$

(9) $$\frac{1}{B(n, m)} = m \binom{n+m-1}{n-1} = n \binom{n+m-1}{m-1} \qquad n, m, \text{ positive integers.}$$

1.5.1. Definite integrals expressible in terms of the beta function

By means of suitable substitutions, a number of definite integrals, such as the following, are reducible to the beta function:

(10) $B(x,y) = 2^{1-x-y} \int_0^1 [(1+t)^{x-1}(1-t)^{y-1} + (1+t)^{y-1}(1-t)^{x-1}]\, dt$

$\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0,$

(11) $\int_0^1 t^{x-1}(1-t)^{y-1}(1+bt)^{-x-y}\, dt = (1+b)^{-x} B(x,y)$

$b > -1, \quad \operatorname{Re} x > 0, \quad \operatorname{Re} y > 0,$

(12) $\int_0^\infty t^{x-1}(1+bt)^{-x-y}\, dt = b^{-x} B(x,y)$

$b > 0, \quad \operatorname{Re} x > 0, \quad \operatorname{Re} y > 0,$

(13) $\int_b^a (t-b)^{x-1}(a-t)^{y-1}\, dt = (a-b)^{x+y-1} B(x,y)$

$\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0, \quad b < a,$

(14) $$\int_b^a \frac{(t-b)^{x-1}(a-t)^{y-1}}{(t-c)^{x+y}}\, dt = \frac{(a-b)^{x+y-1}}{(a-c)^x (b-c)^y} B(x,y)$$

$\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0, \quad c < b < a,$

(15) $$\int_b^a \frac{(t-b)^{x-1}(a-t)^{y-1}}{(c-t)^{x+y}}\, dt = \frac{(a-b)^{x+y-1}}{(c-a)^x (c-b)^y} B(x,y)$$

$\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0, \quad b < a < c,$

(16) $\int_0^\infty (1+bt^z)^{-y} t^x\, dt = z^{-1} b^{-(x+1)/z} B[(x+1)/z,\ y-(x+1)/z]$

$z > 0, \quad b > 0, \quad 0 < \operatorname{Re}[(x+1)/z] < \operatorname{Re} y,$

(17) $\int_0^1 t^{x-1}(1-t^z)^{y-1}\, dt = z^{-1} B(xz^{-1}, y)$

$z > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re} x > 0,$

(18) $\int_{-1}^1 (1+t)^{2x-1}(1-t)^{2y-1}(1+t^2)^{-x-y}\, dt = 2^{x+y-2} B(x,y)$

$\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0.$

Substituting trigonometric and hyperbolic functions, we obtain a number of integrals involving trigonometric and hyperbolic functions:

(19) $\int_0^{\frac{1}{2}\pi} (\sin t)^{2x-1}(\cos t)^{2y-1}\, dt = \frac{1}{2} B(x,y)$

$\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0,$

(20) $$\int_0^{\frac{1}{2}\pi} \frac{(\sin t)^{2x-1}(\cos t)^{2y-1}}{(1+b\sin^2 t)^{x+y}}\, dt = \frac{1}{2}(1+b)^{-x} B(x,y)$$

$\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0, \quad b > -1,$

$$(21)\quad \int_0^{\frac{1}{2}\pi} \frac{(\sin t)^{2x-1}(\cos t)^{2y-1}}{(\cos^2 t + b\sin^2 t)^{x+y}}\,dt = \tfrac{1}{2} b^{-x}\,\mathrm{B}(x,y)$$

$\mathrm{Re}\,x > 0,\quad \mathrm{Re}\,y > 0,\quad b > 0,$

$$(22)\quad \int_0^\infty \cosh t\,(\sinh t)^{2x-1}(1 + b\sinh^2 t)^{-x-y}\,dt = \tfrac{1}{2} b^{-x}\,\mathrm{B}(x,y)$$

$\mathrm{Re}\,x > 0,\quad \mathrm{Re}\,y > 0,\quad b > 0,$

$$(23)\quad \int_0^\infty (\sinh t)^\alpha (\cosh t)^{-\beta}\,dt = \tfrac{1}{2}\mathrm{B}(\tfrac{1}{2}\alpha + \tfrac{1}{2},\ \tfrac{1}{2}\beta - \tfrac{1}{2}\alpha)$$

$\mathrm{Re}\,\alpha > -1,\quad \mathrm{Re}(\alpha - \beta) < 0,$

$$(24)\quad \int_0^\infty e^{-xt}(1 - e^{-tz})^{y-1}\,dt = z^{-1}\,\mathrm{B}(x/z, y)$$

$\mathrm{Re}\,x/z > 0,\quad \mathrm{Re}\,z > 0,\quad \mathrm{Re}\,y > 0,$

$$(25)\quad \int_0^\infty e^{-\alpha t}[\sinh(\beta t)]^\gamma\,dt = \beta^{-1}\,2^{-1-\gamma}\,\mathrm{B}(\tfrac{1}{2}\tfrac{\alpha}{\beta} - \tfrac{1}{2}\gamma,\ 1+\gamma)$$

$\mathrm{Re}\,\gamma > -1,\quad \mathrm{Re}\,\beta > 0,\quad \mathrm{Re}(\alpha/\beta) > \mathrm{Re}\,\gamma,$

$$(26)\quad \int_0^\infty \frac{\cosh(2\alpha t)}{[\cosh(pt)]^{2\beta}}\,dt = 4^{\beta-1} p^{-1}\,\mathrm{B}(\beta + \alpha/p,\ \beta - \alpha/p)$$

$\mathrm{Re}(\beta \pm \alpha/p) > 0,\quad p > 0,$

$$(27)\quad \int_0^\infty \cos(2zt)\,\mathrm{sech}(\pi t)\,dt = \tfrac{1}{2}\,\mathrm{sech}\,z \qquad |\mathrm{Im}\,z| < \tfrac{1}{2}\pi,$$

$$(28)\quad \int_0^\infty \cosh(2zt)\,\mathrm{sech}(\pi t)\,dt = \tfrac{1}{2}\sec z \qquad |\mathrm{Re}\,z| < \tfrac{1}{2}\pi.$$

Formula (27) is known as Ramanujan's formula.

Formulas (12), (13), (17), (19) originate from (1); (11) from (2); (10) and (26) from (3); (14), (15), (20), and (21) from (11); (16) and (22) from (12); (18) from (16); (24) from (17); (23) from (22); (25) from (24); (27) and (28) from (26); all are obtained by easily recognizable substitutions or specializations of parameters. Evidently the range of validity of the formulas (11), (20), and (12), (16), (21), (22) with respect to b can be extended to any values of b in the complex b-plane supposed cut along the real axis from -1 to $-\infty$ and from 0 to $-\infty$ respectively.

By complex integration it is possible to express some further trigonometric integrals in terms of the gamma function. Consider

$$\int_C (z^{-1} - z)^\alpha z^{\beta-1}\,dz$$

where C is a contour consisting of the upper semi-circle $|z| = 1$ and its diameter. The contour is indented at $z = 0, \pm 1$, and the radius of each indentation is ϵ. On letting ϵ approach zero, one obtains (cf. Nielsen, 1906, p. 158) the following result:

$$(29)\quad \int_0^{\pi} (\sin t)^{\alpha} e^{i\beta t}\, dt = \frac{\pi}{2^{\alpha}} \frac{\Gamma(1+\alpha)}{\Gamma(1+\frac{\alpha+\beta}{2})\,\Gamma(1+\frac{\alpha-\beta}{2})} e^{i\frac{1}{2}\pi\beta}$$

$\operatorname{Re} \alpha > -1.$

If C is a contour consisting of the semi-circle $|z| = 1$ in the right-half plane and the straight line joining the points $z = \pm i$, with indentations at $z = 0$, $\pm i$, and if the radii of indentation are made to approach 0, the evaluation of

$$\int_C (z^{-1} + z)^{\alpha} z^{\beta-1}\, dz$$

gives

$$(30)\quad \int_0^{\frac{1}{2}\pi} (\cos t)^{\alpha} \cos(\beta t)\, dt = \frac{\pi}{2^{\alpha+1}} \frac{\Gamma(1+\alpha)}{\Gamma(1+\frac{\alpha+\beta}{2})\;\Gamma(1+\frac{\alpha-\beta}{2})}$$

$\operatorname{Re} \alpha > -1.$

For other similar integrals see 2.4(6) to 2.4(10).

Next consider

$$\int_C z^{\alpha-1} e^{-cz}\, dz \qquad c > 0,$$

where the contour C consists of the real axis from $+\epsilon$ to $+R$, the arc of the circle $z = Re^{i\phi}$ from $\phi = 0$ to $\phi = \beta(-\frac{1}{2}\pi \le \beta \le \frac{1}{2}\pi)$, the straight line from $z = Re^{i\beta}$ to $\epsilon e^{i\beta}$, and the arc of the circle $z = \epsilon e^{i\phi}$ from $\phi = \beta$ to $\phi = 0$. Since the value of the contour integral is zero, on making $\epsilon \to 0$ and $R \to \infty$ it follows that

$$(31)\quad \int_0^{\infty} t^{\alpha-1} e^{-ct\cos\beta - ict\sin\beta}\, dt = \Gamma(\alpha)\, c^{-\alpha} e^{-i\alpha\beta}$$

$-\frac{1}{2}\pi < \beta < \frac{1}{2}\pi$, $\operatorname{Re} \alpha > 0$, or $\beta = \pm\frac{1}{2}\pi$, $0 < \operatorname{Re} \alpha < 1.$

With $p = c \cos\beta$, $q = c \sin\beta$

$$(32)\quad \int_0^{\infty} t^{\alpha-1} e^{-pt-iqt}\, dt = \Gamma(\alpha)\,(p^2 + q^2)^{-\frac{1}{2}\alpha} e^{-i\alpha\tan^{-1}(p/q)^{-1}}$$

$p > 0$, $\operatorname{Re} \alpha > 0$, or $p = 0$, $0 < \operatorname{Re} \alpha < 1.$

With $p + iq = s$, $\tan^{-1}(q/p) = \arg s$

$$(33)\quad \int_0^{\infty} t^{\alpha-1} e^{-st}\, dt = \Gamma(\alpha)\, s^{-\alpha}$$

$\operatorname{Re} \alpha > 0$, $\operatorname{Re} s > 0$ or $\operatorname{Re} s = 0$, $0 < \operatorname{Re} \alpha < 1,$

and hence more generally

(34) $\int_0^{\infty e^{i\delta}} t^{\alpha-1} e^{-st} dt = \Gamma(\alpha) s^{-\alpha}$

$\text{Re}\, \alpha > 0, \quad -(\tfrac{1}{2}\pi + \delta) < \arg s < \tfrac{1}{2}\pi - \delta.$

From (32) one obtains

(35) $\int_0^\infty t^{\alpha-1} e^{-ct\cos\beta} \cos(ct \sin\beta)\, dt = \Gamma(\alpha)\, c^{-\alpha} \cos(\alpha\beta)$

$c > 0, \quad \text{Re}\, \alpha > 0, \quad -\tfrac{1}{2}\pi < \beta < \tfrac{1}{2}\pi,$

(36) $\int_0^\infty t^{\alpha-1} e^{-ct\cos\beta} \sin(ct \sin\beta)\, dt = \Gamma(\alpha)\, c^{-\alpha} \sin(\alpha\beta)$

$c > 0, \quad \text{Re}\, \alpha > -1, \quad -\tfrac{1}{2}\pi < \beta < \tfrac{1}{2}\pi.$

If β approaches $\tfrac{1}{2}\pi$ and c is greater than zero, then

(37) $\int_0^\infty t^{\alpha-1} \cos(ct)\, dt = c^{-\alpha}\Gamma(\alpha) \cos(\tfrac{1}{2}\pi\alpha) \qquad 0 < \text{Re}\, \alpha < 1,$

(38) $\int_0^\infty t^{\alpha-1} \sin(ct)\, dt = c^{-\alpha}\Gamma(\alpha) \sin(\tfrac{1}{2}\pi\alpha) \qquad -1 < \text{Re}\, \alpha < 1.$

Furthermore, one obtains

(39) $\int_0^\infty \cos(at^p)\, dt = (pa^{1/p})^{-1}\, \Gamma(1/p) \cos[\pi(2p)^{-1}] \qquad a > 0, \quad p > 1,$

(40) $\int_0^\infty \sin(at^p)\, dt = (pa^{1/p})^{-1}\, \Gamma(1/p) \sin[\pi(2p)^{-1}]$

1.6. The gamma and beta functions expressed as contour integrals

We use the notation $\int_\zeta^{(0+)} f(t)\, dt$ for an integral taken along a contour C which starts at a point ζ, encircles the origin once counter-clockwise and returns to its starting point, it being understood that all singularities of the integrand except $t = 0$ are outside C.

Consider $\int_{-\infty}^{(0+)} e^t t^{-z}\, dt$, the initial and final values of $\arg t$ being $-\pi$ and $+\pi$ respectively. Taking C to consist of the lower edge of the cut from $-\infty$ to $-\rho$, the circle $t = \rho e^{i\varphi}$ $(-\pi \le \varphi \le \pi)$, and the upper edge of the cut from $-\rho$ to $-\infty$, we find that

$$\int_{-\infty}^{(0+)} e^t t^{-z}\, dt = 2i \sin(\pi z) \int_\rho^\infty e^{-v} v^{-z}\, dv + I,$$

where I denotes the integral along the circle $|t| = \rho$. Since I tends to zero with ρ, provided that $\text{Re}\, z < 1$, we have, in view of 1.1(1),

(1) $\int_{-\infty}^{(0+)} e^t t^{-z}\, dt = 2i \sin(\pi z)\, \Gamma(1-z)$

or, by means of 1.2(6), Hankel's representation

(2) $1/\Gamma(z) = 1/(2\pi i) \int_{-\infty}^{(0+)} e^t t^{-z}\, dt \qquad |\arg t| \le \pi.$

Since both sides of this equation represent entire functions of z, the equation is valid for all values of z.

If we replace z by $1-z$ in (1) we obtain

$$(3) \quad 2i\,\sin(\pi z)\,\Gamma(z) = \int_{-\infty}^{(0+)} e^{t}\, t^{z-1}\,dt \qquad |\arg t| \le \pi.$$

Equation (3) may be written as

$$(4) \quad 2i\,\sin(\pi z)\,\Gamma(z) = -\int_{\infty}^{(0+)} (-t)^{z-1}\,e^{-t}dt \qquad |\arg(-t)| \le \pi.$$

In the same manner, a more general expression can be obtained by the aid of 1.5(34) if we consider the contour integral

$$\int_{\infty \exp i\delta}^{(0+)} t^{s-1}\,e^{-t\zeta}\,dt.$$

The initial and final values of $\arg t$ are now taken to be δ and $2\pi+\delta$. This leads to

$$(5) \quad \Gamma(s) = \zeta^{s}\,(e^{2\pi i s}-1)^{-1} \int_{\infty \exp i\delta}^{(0+)} t^{s-1}\,e^{-t\zeta}\,dt$$

$$-(\tfrac{1}{2}\pi+\delta) < \arg\zeta < \tfrac{1}{2}\pi-\delta, \quad \delta \le \arg t \le 2\pi+\delta, \quad s \ne 0, \pm 1, \pm 2, \ldots,$$

or, by replacing s by $1-s$ and using 1.2(6), we have

$$(6) \quad 2\pi i\,(\zeta e^{-i\pi})^{s-1}/\Gamma(s) = \int_{\infty \exp i\delta}^{(0+)} t^{-s}\,e^{-t\zeta}\,dt$$

$$-(\tfrac{1}{2}\pi+\delta) < \arg\zeta < \tfrac{1}{2}\pi-\delta, \quad \delta \le \arg t \le 2\pi+\delta,$$

which is valid for all values of s.

Finally, consider

$$\int_c t^{x-1}\,(1-t)^{y-1}\,dt = \int_c f(t)\,dt$$

taken around a closed contour which starts from a point A on the real t-axis between 0 and 1, and consists of a loop around $t=1$ in the positive sense, a loop around $t=0$ in the positive sense, a loop around $t=1$ in the negative sense, and a loop around $t=0$ in the negative sense, so that $f(t)$ returns to A with its initial value, which is positive real and taken with argument zero. Take the loop around 1 to consist of the line from A to $1-\rho$, the small circle $|t-1| = \rho$ and the line from $1-\rho$ to A, and similarly with the other loops. On making $\rho \to 0$,

$$\int^{(1+,0+,1-,0-)} t^{x-1}\,(1-t)^{y-1}\,dt = (1-e^{2\pi i x})\,(1-e^{2\pi i y})\,\mathrm{B}(x,y)$$

$$\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0.$$

Hence

$$(7) \quad \mathrm{B}(x,y) = \frac{-e^{-i\pi(x+y)}}{4\sin(\pi x)\,\sin(\pi y)} \int^{(1+,0+,1-,0-)} t^{x-1}\,(1-t)^{y-1}\,dt,$$

and by the theory of analytic continuation this formula, derived in the first instance for Re $x > 0$, Re $y > 0$, x, y, not integers, holds for all values of x and y except integers. It is due to Pochhammer.

In a similar manner $B(x, y)$ can be represented as a single loop integral

(8) $$B(x, y) = \tfrac{1}{2}\operatorname{csch}(\pi i y) \int_0^{(1+)} t^{x-1} (t-1)^{y-1}\, dt$$
$$\operatorname{Re} x > 0, \quad |\arg(t-1)| \leq \pi, \quad y \neq 0, \pm 1, \pm 2, \ldots,$$

(9) $$B(x, y) = -\tfrac{1}{2}\operatorname{csch}(\pi i x) \int_1^{(0+)} (-t)^{x-1} (1-t)^{y-1}\, dt$$
$$\operatorname{Re} y > 0, \quad |\arg(-t)| \leq \pi, \quad x \neq 0, \pm 1, \pm 2, \ldots .$$

1.7. The ψ function

The function $\psi(z)$ is the logarithmic derivative of the gamma function:

(1) $$\psi(z) = \frac{d \log \Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log \Gamma(z) = \int_1^z \psi(z)\, dz.$$

From equations 1.1 (2) and 1.1 (3) we obtain the representations

(2) $$\psi(z) = \lim_{n \to \infty} \left[\log n - \frac{1}{z} - \frac{1}{z+1} - \frac{1}{z+2} - \cdots - \frac{1}{z+n} \right],$$

(3) $$\psi(z) = -\gamma - (1/z) + \sum_{n=1}^{\infty} z/[n(z+n)]$$
$$= -\gamma + (z-1) \sum_{n=0}^{\infty} 1/[(n+1)(z+n)].$$

The ψ function is meromorphic with simple poles at $z = 0, -1, -2, \ldots$. Clearly

(4) $$\psi(1) = -\gamma.$$

From equation 1.3 (2) with $u = z$, $v = 1$ we have

(5) $$\log \frac{\Gamma(1+z)}{\Gamma(z)} = \log z = -\gamma + \sum_{n=0}^{\infty} \{(n+1)^{-1} - \log[1 + 1/(n+z)]\}.$$

From equations (3) and (5)

(6) $$\psi(z) = \log z - \sum_{n=0}^{\infty} \{(n+z)^{-1} - \log[1 + 1/(n+z)]\}.$$

From equation (6) and 1.1 (1)

(7) $$\gamma = -\psi(1) = \sum_{n=1}^{\infty} [n^{-1} - \log(1 + n^{-1})] = -\int_0^{\infty} e^{-t} \log t\, dt.$$

1.7.1. Functional equations for $\psi(z)$

From equations 1.2(1), 1.2(2), 1.2(6) and 1.2(11) we have

(8) $\psi(z) = \psi(1+z) - 1/z,$

(9) $\psi(1+n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \gamma,$

(10) $\psi(z+n) = \frac{1}{z} + \frac{1}{z+1} + \cdots + \frac{1}{z+n-1} + \psi(z) \qquad n = 1, 2, 3, \ldots,$

(11) $\psi(z) - \psi(1-z) = -\pi \operatorname{ctn}(\pi z),$

$\psi(z) - \psi(-z) = -\pi \operatorname{ctn}(\pi z) - 1/z,$

$\psi(1+z) - \psi(1-z) = z^{-1} - \pi \operatorname{ctn}(\pi z),$

$\psi(\tfrac{1}{2}+z) - \psi(\tfrac{1}{2}-z) = \pi \tan(\pi z),$

(12) $\psi(mz) = m^{-1} \sum_{r=0}^{m-1} \psi(z + r/m) + \log m.$

1.7.2. Integral representations for $\psi(z)$

The formula

(13) $\psi(z) = -\gamma + \int_0^1 (1 - t^{z-1})(1-t)^{-1}\, dt \qquad \operatorname{Re} z > 0,$

is easily verified by expanding $(1-t)^{-1}$ into a series, integrating term by term, and using (3).

The substitution $t = e^{-x}$ gives

(14) $\psi(z) = -\gamma + \int_0^\infty (e^{-t} - e^{-tz})(1 - e^{-t})^{-1}\, dt \qquad \operatorname{Re} z > 0.$

Hence we have

$$\psi[\tfrac{1}{2} + \tfrac{1}{2}(a+\beta)/b] - \psi[\tfrac{1}{2} + \tfrac{1}{2}(a-\beta)/b]$$
$$= 2b \int_0^\infty e^{-at} \sinh(\beta t)[\sinh(bt)]^{-1}\, dt \qquad \operatorname{Re}(a + b \pm \beta) > 0.$$

From (11) we obtain a formula for $\psi(z)$ valid for $\operatorname{Re} z < 1$,

(15) $\psi(z) = -\gamma - \pi \operatorname{ctn}(\pi z) + \int_0^1 (1 - t^{-z})(1-t)^{-1}\, dt \qquad \operatorname{Re} z < 1$

or

(16) $\psi(z) = -\gamma - \pi \operatorname{ctn}(\pi z) + \int_0^\infty (1 - e^{tz})(e^t - 1)^{-1}\, dt \qquad \operatorname{Re} z < 1.$

Gauss' integral formula,

(17) $\psi(z) = \int_0^\infty [t^{-1} e^{-t} - (1 - e^{-t})^{-1} e^{-tz}]\, dt \qquad \operatorname{Re} z > 0,$

can be proved as follows. Integrating

$$x^{-1} = \int_0^\infty e^{-xt}\, dt$$

with respect to x from 1 to n we have

(18) $$\log n = \int_0^\infty (e^{-t} - e^{-nt})\, t^{-1}\, dt.$$

Introducing this and $1/(z+m) = \int_0^\infty e^{-(m+z)t} dt$ in (2) we have

$$\psi(z) = \lim_{n\to\infty} \left\{ \int_0^\infty [(e^{-t} - e^{-nt})\, t^{-1} - e^{-tz} - e^{-t(z+1)} - \cdots - e^{-t(z+n)}]\, dt \right\}$$

$$= \lim_{n\to\infty} \left\{ \int_0^\infty [t^{-1} e^{-t} - (1-e^{-t})^{-1} e^{-tz}]\, dt \right.$$

$$\left. - \int_0^\infty e^{-nt} [t^{-1} - (1-e^{-t})^{-1} e^{-t(z+1)}]\, dt \right\}.$$

The first integral is independent of n, and the second tends to zero as $n \to \infty$. This proves (17).

Taking $z = 1$ in (17) an integral formula for Euler's constant is obtained:

(19) $$\gamma = \int_0^\infty [(1-e^{-t})^{-1} - t^{-1}]\, e^{-t}\, dt.$$

With $t = \log(1+x)$ and $\delta = \log(1+\Delta)$ we have from (17)

$$\psi(z) = \lim_{\delta\to 0} \int_\delta^\infty [t^{-1} e^{-t} - (1-e^{-t})^{-1} e^{-zt}]\, dt$$

$$= \lim_{\delta\to 0} \int_\delta^{e^\delta - 1} t^{-1} e^{-t}\, dt + \lim_{\Delta\to 0} \int_\Delta^\infty [e^{-x} - (1+x)^{-z}]\, x^{-1}\, dx.$$

Since the first limit is zero, Dirichlet's formula,

(20) $$\psi(z) = \int_0^\infty [e^{-t} - (1+t)^{-z}]\, t^{-1}\, dt \qquad \operatorname{Re} z > 0,$$

follows. Also we have

(21) $$\gamma = -\psi(1) = -\int_0^\infty [e^{-t} - (1+t)^{-1}]\, t^{-1}\, dt$$

$$= -\int_0^\infty [\cos t - (1+t^2)^{-1}]\, t^{-1}\, dt.$$

The first integral follows from equation (20), and the second can be obtained by integrating $t^{-1} e^{-t} - t^{-1}(1+t)^{-1}$ around a quadrant of a circle indented at the origin, the origin being the centre of the circle.

From equations (20) and (21) we obtain

$$\psi(z) = -\gamma + \int_0^\infty [(1+t)^{-1} - (1+t)^{-z}]\, t^{-1}\, dt \qquad \operatorname{Re} z > 0.$$

Binet's expressions,

(22) $\psi(z) = \log z + \int_0^\infty [t^{-1} - (1-e^{-t})^{-1}]\, e^{-tz} dt$ Re $z > 0$,

(23) $\psi(z) = \log z - \frac{1}{2} z^{-1} - \int_0^\infty [(1-e^{-t})^{-1} - t^{-1} - \frac{1}{2}]\, e^{-tz}\, dt$

Re $z > 0$,

(24) $\psi(z) = \log z + \int_0^\infty [(1-e^{t})^{-1} + t^{-1} - 1]\, e^{-tz}\, dt$ Re $z > 0$,

(25) $\psi(z) = \log z - \frac{1}{2} z^{-1} - \int_0^\infty [(e^{t}-1)^{-1} - t^{-1} + \frac{1}{2}]\, e^{-tz}\, dt$ Re $z > 0$,

can easily be obtained from (17) and (18).

The more general expression

(26) $\psi(z) = \log z - \frac{1}{2} z^{-1} - \int_0^{\infty e^{i\beta}} [(e^{t} - 1)^{-1} - t^{-1} + \frac{1}{2}]\, e^{-tz}\, dt$

$-\frac{1}{2}\pi < \beta < \frac{1}{2}\pi, \quad -(\frac{1}{2}\pi + \beta) < \arg z < (\frac{1}{2}\pi - \beta)$

can be deduced from (25) by integrating

$[(e^{t} - 1)^{-1} - t^{-1} + \frac{1}{2}]\, e^{-tz}$

around a sector indented at the origin, as in the derivation of 1.5(31).

From 1.9(9) we obtain

(27) $\psi(z) = \dfrac{d \log \Gamma(z)}{dz} = \log z - \frac{1}{2} z^{-1} - 2\int_0^\infty (t^2 + z^2)^{-1}\, (e^{2\pi t} - 1)^{-1}\, t\, dt$

Re $z > 0$,

which is likewise due to Binet. Hence we have

(28) $\gamma = -\psi(1) = \frac{1}{2} + 2\int_0^\infty (t^2 + 1)^{-1}\, (e^{2\pi t} - 1)^{-1}\, t\, dt.$

1.7.3. The theorem of Gauss

Taking $z = p/q$ in (13), $0 < p < q$, p and q integers, and putting $t = v^q$ we obtain

$$\psi(p/q) = -\gamma + \int_0^1 R(v)\, dv, \quad R(v) = q(v^{p-1} - v^{q-1})(v^q - 1)^{-1}.$$

Since

$$v^q - 1 = (v-1) \prod_{n=1}^{q-1} [v - \exp(2\pi i n/q)],$$

we can decompose $R(v)$ into partial fractions:

$$R(v) = \sum_{n=1}^{q-1} [\exp(2\pi i p n/q) - 1]\, [v - \exp(2\pi i n/q)]^{-1}.$$

Introducing $R(v)$ and integrating it can be shown (Böhmer 1939, p. 77) that

$$(29)\quad \psi(p/q) = -\gamma - \log q - \tfrac{1}{2}\pi \operatorname{ctn}(\pi p/q) + \sum_{n=1}^{\leq \frac{1}{2} q}{}' \cos(2\pi p n/q) \log[2 - 2\cos(2\pi n/q)].$$

The prime indicates that in case of an even q only one-half of the last term shall be taken in the sum. Thus for a positive proper fraction z the value of $\psi(z)$ can be expressed as a finite combination of elementary functions. By means of (10) this result may be extended to every rational value of z. This is Gauss' theorem.

1.7.4. Some infinite series connected with the ψ function

If we define

$$\Delta f(z) = f(z+1) - f(z), \qquad \Delta^n f(z) = \sum_{m=0}^{n} \binom{n}{m} f(z+n-m)(-1)^m,$$

it follows from 1.7(8) that

$$\Delta \psi(a+z) = 1/(a+z),$$

so that we have

$$\Delta^2 \psi(a+z) = \Delta[1/(a+z)] = -1/[(a+z)(a+z+1)]$$

and

$$\Delta^n \psi(a+z) = \Delta^{n-1}[1/(a+z)] = (-1)^{n-1}(n-1)!/[(a+z)(a+z+1)\cdots(a+z+n-1)].$$

Hence the development of $\psi(a+z)$ in a factorial series is convergent for $\operatorname{Re}(a+z) > 0$, a not a negative integer, and is of the form (Nörlund 1924, p. 261)

$$(30)\quad \psi(a+z) = \psi(a) + \frac{z}{a} - \frac{1}{2}\,\frac{z(z-1)}{a(a+1)} + \frac{1}{3}\,\frac{z(z-1)(z-2)}{a(a+1)(a+2)} - \cdots$$

The functional equation 1.7(10) is useful for summing some series. We have for instance:

$$(31)\quad \sum_{m=0}^{n} (a+mb)^{-1} = b^{-1} \sum_{m=0}^{n} (m+a/b)^{-1} = b^{-1}[\psi(n+1+a/b) - \psi(a/b)],$$

$$(32)\quad \frac{1}{a+b} - \frac{1}{a+2b} + \cdots - \frac{1}{a+2nb} = \frac{1}{4b}\sum_{m=1}^{n}\left(\frac{a-b}{2b}+m\right)^{-1}\left(\frac{a}{2b}+m\right)^{-1}$$

$$= \tfrac{1}{2} b^{-1}\left[\psi\left(\frac{a}{2b}+n+\frac{1}{2}\right) - \psi\left(\frac{a}{2b}+\frac{1}{2}\right) - \psi\left(\frac{a}{2b}+n+1\right) + \psi\left(\frac{a}{2b}+1\right)\right],$$

and, if $n \to \infty$,

(33) $$\frac{1}{a+b} - \frac{1}{a+2b} + \frac{1}{a+3b} - \cdots = \tfrac{1}{2} b^{-1}[\psi(1+\tfrac{1}{2} ab^{-1}] - \psi(\tfrac{1}{2} + \tfrac{1}{2} ab^{-1})].$$

1.8. The function $G(z)$

The function $G(z)$ is defined by

(1) $$G(z) = \psi(\tfrac{1}{2} + \tfrac{1}{2} z) - \psi(\tfrac{1}{2} z).$$

From 1.7(13) and 1.7(14) we have

(2) $$G(z) = 2\int_0^1 t^{z-1}(1+t)^{-1}\,dt \qquad \operatorname{Re} z > 0,$$

(3) $$G(z) = 2\int_0^\infty e^{-zt}(1+e^{-t})^{-1}\,dt \qquad \operatorname{Re} z > 0.$$

A consideration of $\int_c e^{-zt}(1+e^{-t})^{-1}\,dt$ extended over the contour used in deriving 1.5(31) yields the more general representation

(4) $$G(z) = 2\int_0^{\infty e^{i\beta}} e^{-zt}(1+e^{-t})^{-1}\,dt$$
$$-\tfrac{1}{2}\pi < \beta < \tfrac{1}{2}\pi, \quad -(\tfrac{1}{2}\pi + \beta) < \arg z < \tfrac{1}{2}\pi - \beta,$$

or

(5) $$G(z) = z^{-1} + \int_0^{\infty e^{i\beta}} \tanh(\tfrac{1}{2} t)\, e^{-zt}\,dt$$
$$-\tfrac{1}{2}\pi < \beta < \tfrac{1}{2}\pi, \quad -(\tfrac{1}{2}\pi + \beta) < \arg z < \tfrac{1}{2}\pi - \beta.$$

If we expand $(1+t)^{-1}$ in (2), and integrate term by term, we obtain

(6) $$G(z) = 2\sum_{n=0}^{\infty} (-1)^n (z+n)^{-1} = 2z^{-1}\,{}_2F_1(1, z; 1+z; -1).$$

The functional equations

(7) $$G(1+z) = 2z^{-1} - G(z),$$

(8) $$G(1-z) = 2\pi \csc(\pi z) - G(z),$$

(9) $$G(mz) = -(2/m)\sum_{r=0}^{m-1} (-1)^r \psi(z + r/m) \qquad m \text{ even},$$

(10) $$G(mz) = (1/m)\sum_{r=0}^{m-1} (-1)^r G(z + r/m) \qquad m \text{ odd},$$

follow from (1) in conjunction with 1.7(1).

1.9. Expressions for the function $\log \Gamma(z)$

From 1.7(17) we obtain Malmstén's formula

(1) $\log \Gamma(z) = \int_1^z \psi(z)\, dz = \int_0^\infty \left[(z-1) - \frac{1-e^{-(z-1)t}}{1-e^{-t}} \right] \frac{e^{-t}}{t}\, dt$ $\qquad \operatorname{Re} z > 0$

and from 1.7(25)

(2) $\log \Gamma(z) = (z - \tfrac{1}{2}) \log z - z + 1$

$$+ \int_0^\infty [(e^t - 1)^{-1} - t^{-1} + \tfrac{1}{2}] (e^{-tz} - e^{-t})\, t^{-1}\, dt \qquad \operatorname{Re} z > 0.$$

Since (Whittaker-Watson, 1927, p. 249)

(3) $\int_0^\infty [\tfrac{1}{2} - t^{-1} + (e^t - 1)^{-1}]\, t^{-1} e^{-t} dt = 1 - \tfrac{1}{2} \log(2\pi),$

we have Binet's first expression of $\log \Gamma(z)$,

(4) $\log \Gamma(z) = (z - \tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log(2\pi)$

$$+ \int_0^\infty [(e^t - 1)^{-1} - t^{-1} + \tfrac{1}{2}]\, t^{-1} e^{-tz}\, dt \qquad \operatorname{Re} z > 0,$$

or, more generally [cf. 1.5(1) and also 1.7(25), 1.7(26)],

(5) $\log \Gamma(z) = (z - \tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log(2\pi)$

$$+ \int_0^{\infty e^{i\beta}} [(e^t - 1)^{-1} - t^{-1} + \tfrac{1}{2}]\, t^{-1} e^{-tz}\, dt$$

$$-\tfrac{1}{2}\pi < \beta < \tfrac{1}{2}\pi, \quad -(\tfrac{1}{2}\pi + \beta) < \arg z < \tfrac{1}{2}\pi - \beta.$$

From 1.2(6) one obtains

$$\log \Gamma(z) = \log \pi - \log(\sin \pi z) - \log \Gamma(1 - z)$$

and hence

(6) $\log \Gamma(z) = \log \pi - \log(\sin \pi z)$

$$- \int_0^\infty [(e^{zt} - 1)(1 - e^{-t})^{-1} - z]\, t^{-1} e^{-t} dt \qquad \operatorname{Re} z < 1.$$

Adding (1) and (6) we have

(7) $\log \Gamma(z) = \tfrac{1}{2} \log \pi - \tfrac{1}{2} \log(\sin \pi z)$

$$+ \tfrac{1}{2} \int_0^\infty \{\sinh[(\tfrac{1}{2} - z)t] \operatorname{csch}(t/2) - (1 - 2z) e^{-t}\}\, t^{-1}\, dt$$

$$0 < \operatorname{Re} z < 1.$$

Since

$$|[\tfrac{1}{2} - t^{-1} + (e^t - 1)^{-1}]\, t^{-1}| \le K \quad \text{for} \quad 0 \le t < \infty,$$

it is easily seen from Binet's first expression (4) that

(8) $|\log \Gamma(z) - (z - \tfrac{1}{2}) \log z + z - \tfrac{1}{2} \log 2\pi| < K/x \qquad z = x + iy.$

Finally, we derive Binet's second expression for $\log \Gamma(z)$

(9) $$\log \Gamma(z) = (z - \tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log(2\pi) + 2 \int_0^\infty \frac{\tan^{-1}(t/z)}{e^{2\pi t} - 1}\, dt$$

Re $z > 0$.

From 1.7(3) we have

(10) $$\psi'(z) = \frac{d^2 \log \Gamma(z)}{dz^2} = \sum_{n=0}^{\infty} 1/(z+n)^2.$$

Now we make use of a summation formula due to Plana (Lindelöf, 1906, p. 61),

(11) $$\sum_{n=0}^{\infty} f(n) = \tfrac{1}{2} f(0) + \int_0^\infty f(\tau)\, d\tau + i \int_0^\infty [f(it) - f(-it)](e^{2\pi t} - 1)^{-1}\, dt,$$

valid if

1) $f(\zeta)$ is regular for Re $\zeta \geq 0$, $\zeta = \tau + it$,

2) $\lim_{t \to \infty} e^{-2\pi |t|} f(\tau + it) = 0$ uniformly for $0 \leq \tau < \infty$,

3) $\lim_{\tau \to \infty} \int_{-\infty}^{\infty} e^{-2\pi |t|} \, |f(\tau + it)| \, dt = 0.$

Taking $f(\zeta) = 1/(z+\zeta)^2$ (Re $z > 0$) in (11) we find that

(12) $$\sum_{n=0}^{\infty} 1/(z+n)^2 = \psi'(z) = \tfrac{1}{2} z^{-2} + z^{-1}$$
$$+ \int_0^\infty 4tz(t^2 + z^2)^{-2}(e^{2\pi t} - 1)^{-1}\, dt.$$

Integrating twice from 1 to z we obtain

(13) $$\log \Gamma(z) = (z - \tfrac{1}{2}) \log z + z(A - 1) + B$$
$$+ 2 \int_0^\infty (e^{2\pi t} - 1)^{-1} \tan^{-1}(t/z)\, dt,$$

A and B being integration constants. To determine these, we note that $0 \leq \tan^{-1} x \leq x$ for $x \geq 0$ so that

$$|\log \Gamma(z) - (z - \tfrac{1}{2}) \log z - (A - 1) z - B| < (2/z) \int_0^\infty (e^{2\pi t} - 1)^{-1}\, t\, dt$$

for z real and positive. The right-hand side vanishes as $z \to \infty$ through positive real values, and by comparison with (8) we at once have $A = 0$, $B = \tfrac{1}{2} \log(2\pi)$. This proves (9).

1.9.1. Kummer's series for $\log \Gamma(z)$

The function $\log\Gamma(x)$, $0 < x < 1$, can be expanded in a Fourier series. We shall use the known Fourier expansions (Bromwich, 1947, pp. 356, 393, and 370 respectively):

$$\log(\sin \pi x) = -\log 2 - \sum_{n=1}^{\infty} (1/n) \cos(2\pi n x),$$

$$\operatorname{csch}(\tfrac{1}{2}t) \sinh(\tfrac{1}{2} - x)t = 8\pi \sum_{n=1}^{\infty} [n \sin(2\pi n x)]/(t^2 + 4\pi^2 n^2),$$

$$\pi(1 - 2x) = 2 \sum_{n=1}^{\infty} (1/n) \sin(2\pi n x).$$

If these are substituted in (7) with $z = x$, we have to evaluate the integral

$$\int_0^\infty \left(\frac{2\pi n}{t^2 + 4\pi^2 n^2} - \frac{e^{-t}}{2\pi n}\right)\frac{dt}{t} = \frac{1}{2\pi n}\int_0^\infty\left(\frac{1}{1+t^2} - e^{-2\pi n t}\right)\frac{dt}{t}$$

$$= \frac{1}{2\pi n}\left[\int_0^\infty\left(\frac{1}{1+t^2} - \cos t\right)\frac{dt}{t} + \int_0^\infty \frac{e^{-t} - e^{-2\pi n t}}{t}\,dt + \int_0^\infty (\cos t - e^{-t})\frac{dt}{t}\right]$$

and by means of 1.7(21) and 1.7(18) this is $(2\pi n)^{-1}[\gamma + \log(2\pi n)]$ since we have for the third integral:

$$\lim_{\delta \to 0} \int_\delta^\infty (\cos t - e^{-t})\, t^{-1}\, dt = \lim_{\delta \to 0} [\operatorname{Ei}(-\delta) - \operatorname{Ci}(\delta)] = 0.$$

Thus we have

(14) $$\log \Gamma(x) = \tfrac{1}{2} \log(2\pi) + \sum_{n=1}^{\infty} [(2n)^{-1} \cos(2\pi n x) + (\gamma + \log 2\pi n)(\pi n)^{-1} \sin(2\pi n x)],$$

$$\log \Gamma(x) = (\tfrac{1}{2} - x)(\gamma + \log 2) + (1 - x)\log \pi - \tfrac{1}{2}\log(\sin \pi x) + \sum_{n=1}^{\infty} (\pi n)^{-1} \log n \sin(2\pi n x) \qquad 0 < x < 1,$$

which is Kummer's series.

A similar representation for $\psi(x)$ is due to Lerch (Nielsen, 1906, p. 204),

(15) $$\psi(x) \sin(\pi x) = -\tfrac{1}{2}\pi \cos(\pi x) - (\gamma + \log 2\pi) \sin \pi x + \sum_{n=1}^{\infty} \log\left(\frac{n}{n+1}\right) \sin(2n+1)\pi x \qquad 0 < x < 1.$$

From (14) we obtain the integral formulas:

(16) $\int_0^1 \log\Gamma(x)\,\sin(2\pi n x)\,dx = \dfrac{\gamma + \log(2\pi n)}{2\pi n}$ $\qquad n = 1, 2, 3, \ldots,$

(17) $\int_0^1 \log\Gamma(x)\,\cos(2\pi n x)\,dx = \dfrac{1}{4n}$ $\qquad n = 1, 2, 3, \ldots,$

(18) $\int_0^1 \log\Gamma(x)\,dx = \tfrac{1}{2}\log(2\pi).$

Furthermore, we have

(19) $\int_x^{x+1} \log\Gamma(t)\,dt = x\log x - x + \tfrac{1}{2}\log(2\pi).$

This formula can be proved in the following way.

From the multiplication formula 1.2(11) we have

$$m^{-1}\log\left[\Gamma(mx)(2\pi)^{m/2-1/2}m^{1/2-mx}\right] = \sum_{r=0}^{m-1} m^{-1}\log\Gamma(x + r/m).$$

If we now let $m \to \infty$, replace $\Gamma(mx)$ by its asymptotic expression 1.18(1), and observe that

$$\lim_{m\to\infty} \sum_{r=0}^{m-1} m^{-1}\log\Gamma(x + r/m) = \int_0^1 \log\Gamma(x+y)\,dy = \int_x^{x+1}\log\Gamma(t)\,dt,$$

we obtain (19).

Replacing x by $x+1, x+2, x+3, \ldots, x+n-1$, respectively, in (19) and adding the equations, we have more generally

(20) $\int_x^{x+n}\log\Gamma(x)\,dx = x\log x + (x+1)\log(x+1) + \cdots$

$$+ (x+n-1)\log(x+n-1) - nx - \tfrac{1}{2}n(n-1) + \tfrac{1}{2}n\log(2\pi)$$

$n = 1, 2, 3, \ldots.$

1.10. The generalized zeta function

see errata The generalized zeta function is defined for Re $s >$ ~~0~~ 1 by the equation

(1) $\zeta(s, v) = \sum_{n=0}^{\infty} (v+n)^{-s}$ $\qquad v \neq 0, -1, -2, \ldots.$

It satisfies the functional equation

(2) $\gamma(s, v) = \gamma(s, m+v) + \sum_{n=0}^{m-1} (n+v)^{-s}$ $\qquad m = 1, 2, 3, \ldots.$

Since for Re $s > 0$ and Re $v > 0$ we have from 1.1(5)

$$(v+n)^{-s}\,\Gamma(s) = \int_0^{\infty} e^{-(v+n)t}\,t^{s-1}\,dt,$$

it follows that

(3) $\Gamma(s)\ \zeta(s, v) = \int_0^\infty t^{s-1}\ e^{-vt}(1-e^{-t})^{-1}\ dt$

$= \int_0^1 x^{v-1}\ (1-x)^{-1}(\log\ 1/x)^{s-1}\ dx$ $\qquad$ $\operatorname{Re} s > 1, \quad \operatorname{Re} v > 0.$

Considering $\int_c t^{s-1}\ e^{-vt}(1-e^{-t})^{-1}\ dt$ taken around the complete boundary of a sector of a circle, indented at the origin [cf. 1.5(1)], we have the more general representation

(4) $\Gamma(s)\ \zeta(s, v) = \int_0^{\infty e^{i\beta}} t^{s-1}\ e^{-vt}(1-e^{-t})^{-1}\ dt$

$\operatorname{Re} s > 1, \quad -\tfrac{1}{2}\pi < \beta < \tfrac{1}{2}\pi, \quad -(\tfrac{1}{2}\pi + \beta) < \arg v < \tfrac{1}{2}\pi - \beta.$

With the notation of section 1.6 equation (3) can be converted into a contour integral,

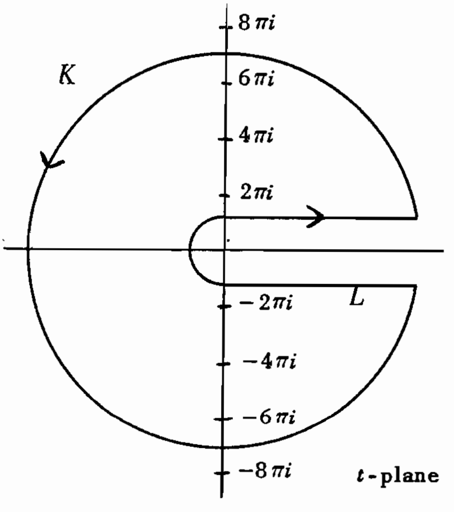
(5) $2\pi i\ \zeta(s, v) = -\Gamma(1-s) \int_\infty^{(0+)} (-t)^{s-1}\ e^{-vt}(1-e^{-t})^{-1}\ dt$

$\operatorname{Re} v > 0, \qquad |\arg(-t)| \leq \pi.$

This integral gives a representation of $\zeta(s, v)$ valid over the whole s-plane with the exception of the points $s = 1, 2, 3, \ldots$. From it Hurwitz' series representation of $\zeta(s, v)$ can be obtained. Consider

$$\int_c (-t)^{s-1}\ e^{-vt}\ (1-e^{-t})^{-1}\ dt$$

taken around a closed contour C starting at the point $t = (2N + 1)\pi$ and



consisting of a circle K and a loop L as indicated in the above figure.

The radius of the circle is $(2N + 1)\pi$ (N an integer), and the loop L does not contain any of the points $t = \pm 2\pi i, \pm 4\pi i, \pm 6\pi i, \ldots$. In the region bounded by C the integrand of (5) is analytic and one valued except at the simple poles $\pm 2\pi i, \pm 4\pi i, \ldots, \pm 2N\pi i$. By the theorem of residues

$$\int_K \frac{(-t)^{s-1} e^{-vt}}{1-e^{-t}}\, dt + \int_L \frac{(-t)^{s-1} e^{-vt}}{1-e^{-t}}\, dt = 2\pi i \sum_{n=1}^{N} (R_n + R_n')$$

where R_n and R_n' are the residues of the integrand respectively at $2n\pi i$ and $-2n\pi i$,

$$R_n = (2n\pi)^{s-1} e^{-i\frac{1}{2}\pi(s-1)} e^{-2n\pi vi}, \quad R_n' = (2n\pi)^{s-1} e^{i\frac{1}{2}\pi(s-1)} e^{2n\pi vi}.$$

Letting $N \to \infty$ we find that the integral over K tends to zero provided $\operatorname{Re} s < 0$ and $0 < v \le 1$. By means of (5) we thus obtain Hurwitz' formula

$$(6) \quad \zeta(s, v) = 2(2\pi)^{s-1}\Gamma(1-s) \sum_{n=1}^{\infty} n^{s-1} \sin(2\pi n v + \tfrac{1}{2}\pi s)$$

$$\operatorname{Re} s < 0, \quad 0 < v \le 1.$$

Finally, we shall take $f(y) = (y + v)^{-s}$ in Plana's summation formula 1.9(11) to find

$$(7) \quad \zeta(s, v) = \frac{1}{2v^s} + \frac{v^{1-s}}{s-1} + 2\int_0^{\infty} \frac{\sin[s \tan^{-1}(t/v)]}{(v^2+t^2)^{\frac{1}{2}s}} \frac{dt}{e^{2\pi t}-1}$$

$$\operatorname{Re} v > 0,$$

which is Hermite's representation of $\zeta(s, v)$.

From (7) it can be seen that $\zeta(s, v)$ has only one singularity (a simple pole with residue 1) in the finite part of the s-plane. Furthermore we have [cf. 1.7(27)]

$$(8) \quad \zeta(0, v) = \tfrac{1}{2} - v,$$

$$(9) \quad \lim_{s\to 1}\left[\zeta(s, v) - \frac{1}{s-1}\right] = \frac{1}{2v} - \log v + 2\int_0^{\infty} \frac{t}{v^2+t^2} \frac{dt}{e^{2\pi t}-1}$$

$$= -\psi(v) \qquad \operatorname{Re} v > 0.$$

Differentiating (7) with respect to s, then putting $s = 0$, and using 1.9(9) we obtain

$$(10) \quad \left[\frac{d\,\zeta(s, v)}{ds}\right]_{s=0} = \log\Gamma(v) - \tfrac{1}{2}\log(2\pi).$$

In the special case when $s = -m(m = 0, 1, 2, \ldots)$, we have

(11) $$\zeta(-m,v) = -\frac{B_{m+1}(v)}{m+1},$$

where $B_r(v)$ denotes the Bernoulli polynomial [cf. 1.13(3)]. To prove this, we note that if s is an integer, the integrand of (5) is a one-valued function of t, and we may apply Cauchy's theorem. If $s = -m$ $(m = 0, 1, 2, \ldots)$, we have [cf. 1.13(2)]

$$(-t)^{-m-1}\frac{e^{-vt}}{1-e^{-t}} = (-1)^{-m-1}t^{-m-2}\frac{te^{-vt}}{1-e^{-t}}$$

$$= (-1)^{m-1}\sum_{n=0}^{\infty}(-1)^n B_n(v)\frac{t^{n-m-2}}{n!}.$$

Thus the residue of the integrand at $t = 0$ is $\frac{B_{m+1}(v)}{(m+1)!}$, and this proves (11).

1.11. The function $\Phi(z, s, v) = \sum_{n=0}^{\infty}(v+n)^{-s} z^n$

The function

(1) $$\Phi(z, s, v) = \sum_{n=0}^{\infty}(v+n)^{-s} z^n \qquad |z| < 1, \quad v \neq 0, -1, -2, \ldots$$

satisfies the equation

(2) $$y(z, s, v) = z^m y(z, s, m+v) + \sum_{n=0}^{m-1}(v+n)^{-s} z^n$$

$$m = 1, 2, 3, \ldots, \quad v \neq 0, -1, -2, \ldots.$$

Since

$$(v+n)^{-s} z^n = [1/\Gamma(s)]\int_0^{\infty} e^{-vt} t^{s-1}(ze^{-t})^n\, dt$$

$$\operatorname{Re} v > 0, \quad \operatorname{Re} s > 0,$$

From 1.1(5), we have the integral formula

(3) $$\Phi(z, s, v) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}\,dt = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-(v-1)t}}{e^t - z}\,dt$$

$\operatorname{Re} v > 0$ and either $|z| \le 1$, $z \neq 1$, $\operatorname{Re} s > 0$ or $z = 1$, $\operatorname{Re} s > 1$.

If a cut is made from 1 to ∞ along the positive real z-axis, Φ is an analytic function of z in the cut z-plane provided that $\operatorname{Re} s > 0$ and $\operatorname{Re} v > 0$.

Another representation by a definite integral can be obtained from the definition (1) and Plana's summation formula 1.9(11)

(4) $\Phi(z, s, v) = \frac{1}{2} v^{-s} + \int_0^\infty (v+t)^{-s} z^t \, dt$

$$- 2 \int_0^\infty \sin\{t \log z - s \tan^{-1}(t/v)\} (v^2 + t^2)^{-\frac{1}{2}s} (e^{2\pi t} - 1)^{-1} \, dt$$

$\operatorname{Re} v > 0.$

For $z = 1$ we have again Hermite's formula 1.10 (7).

Lipschitz's formula

$$2\Gamma(s) \sum_{n=1}^{\infty} e^{in\theta} (v+n)^{-s} = \int_0^\infty t^{s-1} e^{-vt} (e^{i\theta} - e^{-t}) (\cosh t - \cos\theta)^{-1} \, dt$$

$0 < \theta < 2\pi, \quad \operatorname{Re} s > 0, \quad \operatorname{Re} v > -1$

results from (3) by taking $z = e^{i\theta}$.

Φ can be represented as a contour integral

(5) $2\pi i \, \Phi(z, s, v) = -\Gamma(1-s) \int_\infty^{(0+)} (-t)^{s-1} e^{-vt} (1 - ze^{-t})^{-1} \, dt$

$\operatorname{Re} v > 0, \quad |\arg(-t)| \leq \pi$

assuming, as in the analogous work of 1.6, that the contour does not enclose any of the points $t = \log z \pm 2n\pi i (n = 0, 1, 2, \ldots)$, which are poles of the integrand of (5). Equation (5), for every fixed s which is not a positive integer, defines Φ as an analytic function of z regular in the cut plane, and for every fixed z in the cut plane, Φ as an analytic function of s regular, except possibly at the points $s = 1, 2, 3, \ldots$ (it being understood that $\operatorname{Re} v > 0$).

As in the preceding section our function can be represented by a series. In order to do so, consider

$$\int_C (-t)^{s-1} e^{-vt} (1 - ze^{-t})^{-1} \, dt$$

over the contour C consisting of a circle K of radius $(2N+1)\pi$ (N a positive integer) and a loop L round the origin. The center of the circle in this case is the point $t = \log z \, (z \neq 1)$, and all points $t = \log z \pm 2n\pi i$ $(n = 0, 1, 2, \ldots)$ are to be outside the loop. Letting $N \to \infty$, it is found that the integral over K tends to zero provided $\operatorname{Re} s < 0$ and $0 < v \leq 1$. Therefore

$$\Phi(z, s, v) = \Gamma(1-s) \sum_{n=-\infty}^{\infty} R_n,$$

where $R_n = z^{-1}(-t_n)^{s-1} e^{-(v-1)t_n}$ is the residue of the integrand at the pole $t = t_n = \log z + 2n\pi i$. Thus we have

(6) $\Phi(z, s, v) = z^{-v} \Gamma(1-s) \sum_{n=-\infty}^{\infty} (-\log z + 2n\pi i)^{s-1} e^{2n\pi iv}$

$0 < v \leq 1, \quad \operatorname{Re} s < 0, \quad |\arg(-\log z + 2n\pi i)| \leq \pi.$

Writing

$$\sum_{n=-\infty}^{\infty} (-\log z + 2n\pi i)^{s-1} e^{2n\pi iv} = \sum_{n=0}^{\infty} e^{-2n\pi iv} (-\log z - 2n\pi i)^{s-1}$$

$$+ \sum_{n=1}^{\infty} e^{2n\pi iv} (-\log z + 2n\pi i)^{s-1}$$

and comparing with (1) we obtain at once Lerch's transformation formula for the function $\Phi(z, s, v)$:

(7) $$\Phi(z, s, v) = iz^{-v}(2\pi)^{s-1}\Gamma(1-s)\{e^{-i\pi s/2}\Phi[e^{-2\pi iv}, 1-s, (\log z)/(2\pi i)]$$

$$- e^{i\pi(s/2+2v)}\Phi[e^{2\pi iv}, 1-s, 1-(\log z)/(2\pi i)]\}.$$

If in (6) we use the binomial expansions

$$(-\log z + 2n\pi i)^{s-1}$$
$$= -(2n\pi)^{s-1} ie^{i\pi s/2} \sum_{r=0}^{\infty} (-1)^r \binom{s-1}{r} [(\log z)/(2n\pi)]^r e^{-i\pi r/2},$$

$$(-\log z - 2n\pi i)^{s-1}$$
$$= (2n\pi)^{s-1} ie^{-i\pi s/2} \sum_{r=0}^{\infty} \binom{s-1}{r} [(\log z)/(2n\pi)]^r e^{-i\pi r/2},$$

we find

$$z^v \Phi(z, s, v)/\Gamma(1-s) = [\log(1/z)]^{s-1} + 2 \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} (2\pi n)^{s-1}$$

$$\times\left\{(-1)^r \binom{s-1}{2r} \sin(\tfrac{1}{2}\pi s + 2n\pi v) \, [(\log z)/(2n\pi)]^{2r}\right.$$

$$\left.+ (-1)^r \binom{s-1}{2r+1} \cos(\tfrac{1}{2}\pi s + 2n\pi v)[(\log z)/(2\pi n)]^{2r+1}\right\}.$$

Summing with respect to n by means of Hurwitz' formula 1.10(6) we have

(8) $$\Phi(z, s, v) = \frac{\Gamma(1-s)}{z^v}(\log 1/z)^{s-1} + z^{-v} \sum_{r=0}^{\infty} \zeta(s-r, v) \frac{(\log z)^r}{r!}$$

$$|\log z| < 2\pi, \quad s \neq 1, 2, 3, \ldots, \quad v \neq 0, -1, -2, \ldots.$$

If s is a positive integer $s = m$, we first put $s = m + \epsilon$ and we have from 1.17(11) and 1.10(9)

$$(\log 1/z)^{\epsilon} = 1 + \epsilon \log(\log 1/z) + O(\epsilon^2),$$

$$\Gamma(1-s) = \Gamma(1-m-\epsilon) = \frac{(-1)^m}{(m-1)!} [\epsilon^{-1} - \psi(m)] + O(\epsilon),$$

$$\zeta(1+\epsilon, v) = \epsilon^{-1} - \psi(v) + O(\epsilon).$$

Making $\epsilon \to 0$, we then obtain from (8)

$$(9)\quad \Phi(z, m, v) = z^{-v} \left\{ \sum_{n=0}^{\infty}{}' \zeta(m-n, v) \frac{(\log z)^n}{n!} + \frac{(\log z)^{m-1}}{(m-1)!} [\psi(m) - \psi(v) - \log(\log 1/z)] \right\}$$

$m = 2, 3, 4, \ldots, \quad |\log z| < 2\pi, \quad v \neq 0, -1, -2, \ldots.$

The prime indicates that the term with $n = m - 1$ is to be omitted.

In the case where $s = 1$ we have simply

$$(10)\quad \Phi(z, 1, v) = \sum_{n=0}^{\infty} \frac{z^n}{n+v} = v^{-1}\, {}_2F_1(1, v; 1+v; z) \qquad |z| < 1.$$

From 1.8(6) we see that

$$G(v) = 2\,\Phi(-1, 1, v).$$

If s is a negative integer, $s = -m\,(m = 1, 2, 3, \ldots)$, we can use 1.10(11) in order to express Φ, as given by (8), in terms of Bernoulli's polynomials:

$$(11)\quad \Phi(z, -m, v) = \frac{m!}{z^v} (\log 1/z)^{-m-1} - \frac{1}{z^v} \sum_{r=0}^{\infty} \frac{B_{m+r+1}(v)(\log z)^r}{r!(m+r+1)}$$

$|\log z| < 2\pi.$

Finally from (8) and (10) we deduce

$$(12)\quad \lim_{z\to 1} (1-z)^{1-s} \Phi(z, s, v) = \Gamma(1-s) \qquad \operatorname{Re} s < 1,$$

$$(13)\quad \lim_{z\to 1} \Phi(z, 1, v)/[-\log(1-z)] = 1.$$

The properties of the function

$$(14)\quad F(z, s) = \sum_{n=1}^{\infty} (z^n/n^s) = z\,\Phi(z, s, 1)$$

can easily be deduced from the equations (1) to (13). If $s = -m\,(m = 1, 2, 3, \ldots)$, we find from (11) and 1.13(7) that

$$(15)\quad F(z, -m) = m!\,(\log 1/z)^{-m-1} - \sum_{r=0}^{\infty} \frac{B_{m+r+1}}{(m+r+1)\,r!} (\log z)^r$$

$|\log z| < 2\pi,$

where B_{m+r+1} denotes the Bernoulli number.

From Lerch's transformation 1.11(7) we obtain Joncquière's relation

$$(16)\quad F(z,s)+e^{is\pi}\,F(1/z,s)=\frac{(2\pi)^s}{\Gamma(s)}\,e^{i\pi s/2}\zeta\left(1-s,\frac{\log z}{2\pi i}\right).$$

Furthermore we have

$$(17)\quad F(z,-m)=(-1)^{m+1}\,F(1/z,-m) \qquad m=1,2,3,\ldots,$$

$$(18)\quad F(z,m)+(-1)^m\,F(1/z,m)=-\frac{(2\pi i)}{m!}B_m\left(\frac{\log z}{2\pi i}\right) \qquad m=2,3,4,\ldots.$$

These equations furnish the analytical continuation of the series (14) beyond its circle of convergence $|z| = 1$.

If $F_0(z)$ denotes the principal branch of $F(z)$ in the cut z-plane $[0 < \arg(z-1) < 2\pi]$, the cut being imposed from 1 to ∞ along the real axis, the difference of the values of $F_0(z)$ between a point on the upper edge of the cut and a point on the lower edge of the cut is seen from (16) to be

$$(19)\quad F_0(x,s)-F_0(xe^{2i\pi},s)=\frac{2\pi i}{\Gamma(s)}(\log x)^{s-1}.$$

Hence, if we cross the cut, from the upper half-plane to the lower half-plane, we obtain for the continuation $F_1(z)$ of $F_0(z)$

$$(20)\quad F_1(z)=F_0(z)+2\pi i(\log z)^{s-1}/\Gamma(s).$$

The analogous formula for the inverse process of continuation is

$$(21)\quad F_2(z)=F_0(z)-2\pi i(\log z)^{s-1}/\Gamma(s).$$

(For further discussions of the function $F(z,s)$ see Truesdell, 1945, p. 144.)

1.11.1. Euler's dilogarithm

Euler's dilogarithm is defined by

$$(22)\quad L_2(z)=\sum_{n=1}^{\infty}(z^n/n^2)=-\int_0^z z^{-1}\log(1-z)\,dz=F(z,2),$$

which is a special case of (14).

From (18) we get the equation

$$(23)\quad L_2(z)=-L_2(1/z)-\tfrac{1}{2}(\log z)^2+\pi i\log z+\pi^2/3.$$

If we denote the principal branch of $L_2(z)$ by $L_2^*(z)$ $[0 < \arg(z-1) < 2\pi]$, (19) and (20) show that for any branch

$$L_2(z)=L_2^*(z)+2n\pi i\log z+4m\pi^2 \qquad n,m=0,\pm1,\pm2,\ldots.$$

(For a detailed discussion, see O. Hölder, 1928, p. 312. For other special cases of formula (14) see Ramanujan, 1927, p. 40, 336; Rogers, 1905; and Sandham, 1949.)

1.12. The zeta function of Riemann

Putting $v = 1$ in 1.10(1) we obtain Riemann's zeta-function

(1) $\zeta(s) = \zeta(s, 1) = \Phi(1, s, 1) = \sum_{n=1}^{\infty} (1/n^s)$ Re $s > 1$.

Hence, we have

(2) $\sum_{n=1}^{\infty} [(-1)^{n-1}/n^s] = (1 - 2^{1-s})\, \zeta(s) = \Phi(-1, s, 1)$ Re $s > 0$,

(3) $\sum_{n=0}^{\infty} [1/(2n+1)^s] = (1 - 2^{-s})\, \zeta(s) = 2^{-s}\, \Phi(1, s, \tfrac{1}{2})$ Re $s > 1$.

We therefore have the following integral expressions for $\zeta(s)$ [cf. 1.10(3) and 1.11(3)].

(4) $\Gamma(s)\, \zeta(s) = \int_0^{\infty} t^{s-1} (e^t - 1)^{-1}\, dt = 2^{s-1} \int_0^{\infty} e^{-t} t^{s-1} \operatorname{csch} t\, dt$
Re $s > 1$,

(5) $(1 - 2^{1-s})\, \Gamma(s)\, \zeta(s) = \int_0^{\infty} t^{s-1} (e^t + 1)^{-1}\, dt$

$= 2^{s-1} \int_0^{\infty} e^{-t} t^{s-1} \operatorname{sech} t\, dt$ Re $s > 0$,

(6) $2\, \Gamma(s)\, (1 - 2^{-s})\, \zeta(s) = \int_0^{\infty} t^{s-1} \operatorname{csch} t\, dt$ Re $s > 1$.

From 1.11(1) and 1.11(3) we have

(7) $\zeta(s) = \Phi_z(1, s+1, 0) = [2^{s-1}/\Gamma(s+1)] \int_0^{\infty} t^s (\operatorname{csch} t)^2\, dt$
Re $s > 1$,

(8) $(1 - 2^{1-s})\, \zeta(s) = \Phi_z(-1, s+1, 0) = [2^{s-1}/\Gamma(s+1)] \int_0^{\infty} t^s (\operatorname{sech} t)^2\, dt$
Re $s > -1$.

The following representations of $\zeta(s)$ by means of contour integrals

(9) $2\pi i\, \zeta(s) = -\Gamma(1-s) \int_{\infty}^{(0+)} (-t)^{s-1} (e^t - 1)^{-1}\, dt$

(10) $2\pi i (1 - 2^{1-s})\, \zeta(s) = -\Gamma(1-s) \int_{\infty}^{(0+)} (-t)^{s-1} (e^t + 1)^{-1}\, dt$

(11) $4\pi i (1 - 2^{-s})\, \zeta(s) = -\Gamma(1-s) \int_{\infty}^{(0+)} (-t)^{s-1} \operatorname{csch} t\, dt$

with

$$s \neq 1, 2, 3, \ldots, \qquad |\arg(-t)| \leq \pi$$

follow from 1.10(5) and 1.11(5) by means of (1), (2), and (3). The contour in (9) and (11) contains none of the points $t = \pm 2n\pi i$ and in (10) none of the points $t = (2n-1)\pi i$.

From (1) and 1.10(7) we obtain

(12) $$\zeta(s) = \tfrac{1}{2} + 1/(s-1) + 2\int_0^\infty (1+t^2)^{-s/2}(e^{2\pi t}-1)^{-1}\sin(s\tan^{-1}t)\,dt.$$

Furthermore (Lindelöf, 1905, p. 103) we have

(13) $$\zeta(s) = \frac{2^{s-1}}{s-1} - 2^s \int_0^\infty (1+t^2)^{-s/2}(e^{2\pi t}+1)^{-1}\sin(s\tan^{-1}t)\,dt,$$

(14) $$\zeta(s) = \frac{\pi\, 2^{s-2}}{s-1}\int_0^\infty (1+t^2)^{\frac{1}{2}(1-s)}\frac{\cos[(s-1)\tan^{-1}t]}{[\cosh(\frac{1}{2}\pi t)]^2}\,dt,$$

(15) $$\zeta(s) = \frac{2^{s-1}}{1-2^{1-s}}\int_0^\infty (1+t^2)^{-s/2}\frac{\cos(s\tan^{-1}t)}{\cosh(\frac{1}{2}\pi t)}\,dt.$$

These formulas are due to Jensen. The integrals in (12) to (15) define an analytic function for all values of s.

Other integral representations are (Bruijn, 1937)

$$\zeta(s) = (s-1)^{-1} + \pi^{-1}\sin(\pi s)\int_0^\infty [\log(1+x) - \psi(1+x)]\,x^{-s}\,dx,$$

$$\zeta(1+s) = (\pi s)^{-1}\sin(\pi s)\int_0^\infty \psi'(1+x)\,x^{-s}\,dx$$

$$= \pi^{-1}\sin(\pi s)\int_0^\infty [\psi(1+x)+\gamma]\,x^{-1-s}\,dx,$$

$$\zeta(m+s) = (-1)^{m-1}\frac{\Gamma(s)\sin(\pi s)}{\pi\,\Gamma(m+s)}\int_0^\infty \psi^{(m)}(1+x)\,x^{-s}\,dx$$

$$m = 1, 2, 3, \ldots.$$

These formulas are valid for $0 < \operatorname{Re} s < 1$ and $\psi^{(m)}$ is defined in 1.16(1). Furthermore

$$\zeta(s) = (s-1)^{-1} + \frac{\sin(\pi s)}{\pi(s-1)}\int_0^\infty [\psi'(1+x) - (1+x)^{-1}]\,x^{1-s}\,dx$$

$$0 < \operatorname{Re} s < 2, \qquad s \neq 1.$$

Finally we prove Riemann's representation of $\zeta(s)$,

(16) $$\pi^{-s/2}\Gamma(s/2)\,\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (t^{\frac{1}{2}(1-s)} + t^{s/2})\,t^{-1}\,\omega(t)\,dt$$

where

$$\omega(t) = \sum_{n=1}^{\infty} e^{-n^2 \pi t} = \tfrac{1}{2}[\theta_3(0, it) - 1],$$

θ_3 being the elliptic theta function. The integral in (16) represents an analytic function of s for all values of s.

From 1.1(5) we have

$$\int_0^\infty e^{-n^2\pi t} t^{s/2-1}\, dt = \pi^{-s/2}\Gamma(s/2)\, n^{-s} \qquad \text{Re } s > 0.$$

Hence we obtain

$$\pi^{-s/2}\Gamma(s/2)\ \zeta(s) = \int_0^\infty \omega(t)\, t^{s/2-1}\, dt$$

$$= \int_0^1 \omega(t)\, t^{s/2-1}\, dt + \int_1^\infty \omega(t)\, t^{s/2-1}\, dt.$$

But by means of Jacobi's imaginary transformation of the theta functions (Whittaker and Watson, 1927, § 21.51) we have

$$\omega(t) = -\tfrac{1}{2} + \tfrac{1}{2} t^{-1/2} + t^{-1/2}\, \omega(1/t).$$

Introducing this expression into the integral, we obtain

$$\pi^{-s/2}\Gamma(s/2)\ \zeta(s)$$

$$= -\ (1/s) + 1/(s-1)\ + \int_0^1 \omega(1/t)\, t^{s/2-3/2}\, dt + \int_1^\infty \omega(t)\, t^{s/2-1}\, dt,$$

and substituting $1/t = t'$ in the first integral, we obtain (16). For further integral representations see Ramanujan, 1927, p. 72; Hardy, 1949, pp. 333, 337.

A power series expansion of $\zeta(s)$ is (Hardy, 1912, p. 215; Kluyver, 1927, p. 185)

(17) $$\zeta(s) = (s-1)^{-1} + \gamma + \sum_{n=1}^{\infty} \gamma_n (s-1)^n$$

where

$$\gamma_n = \lim_{m\to\infty} \left[\sum_{l=1}^{m} l^{-1} (\log l)^n - (n+1)^{-1} (\log l)^{n+1} \right].$$ See errata!

Putting $v = 1$ in 1.10(8) to 1.10(11) we obtain

(18) $$\zeta(0) = -\tfrac{1}{2}, \qquad \zeta'(0) = -\tfrac{1}{2}\log(2\pi);$$

(19) $$\lim_{s\to 1} [\,\zeta(s) - 1/(s-1)\,] = -\psi(1) = \gamma,$$

and [cf. 1.13(7)]

(20) $$\zeta(-m) = -\frac{B_{m+1}}{m+1} \qquad m = 1, 2, 3, \ldots,$$

or

$$(21)\quad \zeta(-2m)=0, \qquad \zeta(2m)=(-1)^{m+1}\,(2\pi)^{2m}\,\frac{B_{2m}}{2(2m)!}$$

$m = 1, 2, 3, \ldots,$

$$(22)\quad \zeta[-(2m-1)] = -\frac{B_{2m}}{2m}.$$

Putting $v = 1$ in Hurwitz' equation, we obtain Riemann's functional equation for $\zeta(s)$

$$(23)\quad \zeta(s) = \frac{2\,\Gamma(1-s)}{(2\pi)^{1-s}}\,\sin(\pi s/2)\,\zeta(1-s)$$

or in view of 1.2(6)

$$(24)\quad \zeta(1-s) = (2\pi)^{-s}\,2\,\Gamma(s)\,\cos(\pi s/2)\,\zeta(s).$$

Introducing a new function defined by

$$(25)\quad \xi(s) = \frac{s(s-1)}{2}\,\Gamma(s/2)\,\pi^{-s/2}\zeta(s)$$

we have

$$(26)\quad \xi(1-s) = \xi(s).$$

This function is known as Riemann's ξ function. For asymptotic representations of the zeta function see Hutchinson, 1925; Titchmarsh, 1935, 1936; for numerous other results, Titchmarsh, 1930.

If we consider the function

$$(27)\quad L(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^s} \qquad \text{Re } s > 0,$$

which is similar to the ζ function, we have by means of 1.11(1) and 1.11(3)

$$(28)\quad L(s) = 2^{-s}\,\Phi(-1, s, \tfrac{1}{2}) = \frac{1}{2\,\Gamma(s)}\int_0^{\infty} \frac{t^{s-1}}{\cosh t}\,dt \qquad \text{Re } s > 0.$$

Putting $z = -1$, $v = \frac{1}{2}$ in Lerch's transformation 1.11(7), the following functional equation for $L(s)$ is found:

$$(29)\quad L(1-s) = \left(\frac{2}{\pi}\right)^{s}\,\Gamma(s)\,\sin(\pi s/2)\,L(s).$$

(For further discussions see Lichtenbaum, 1931, p. 641.)

1.13. Bernoulli's numbers and polynomials

The Bernoulli numbers B_n are defined by the equation

$$(1)\quad z(e^{z}-1)^{-1} = \sum_{n=0}^{\infty} B_n\,z^n/n! \qquad |z| < 2\pi,$$

and the Bernoulli polynomials $B_n(x)$ by means of

(2) $$ze^{xz}(e^z - 1)^{-1} = \sum_{n=0}^{\infty} B_n(x)\, z^n/n! \qquad |z| < 2\pi.$$

Since the left-hand side of (2) is

$$\left\{\sum_{r=0}^{\infty} B_r\, z^r/r!\right\}\cdot\left\{\sum_{m=0}^{\infty}[(xz)^m/m!]\right\},$$

Cauchy's rule for multiplying power series gives

(3) $$B_n(x) = x^n + \binom{n}{1} B_1\, x^{n-1} + \cdots + \binom{n}{n-1} B_{n-1}x + \binom{n}{n} B_n$$

$$= \sum_{r=0}^{n} \binom{n}{r} B_r\, x^{n-r},$$

$$B_0(x) = 1, \quad B_1(x) = x - \tfrac{1}{2}, \quad B_2(x) = x^2 - x + 1/6,$$

$$B_3(x) = x^3 - 3/2x^2 + \tfrac{1}{2}x, \quad B_4(x) = x^4 - 2x^3 + x^2 - 1/30, \ldots .$$

Clearly we have

(4) $$B_n(0) = B_n.$$

Differentiating (2) with respect to x and comparing coefficients we obtain

(5) $$B_n'(x) = n\, B_{n-1}(x).$$

From (2) it follows that

$$\sum_{n=0}^{\infty}[B_n(x+1) - B_n(x)]\frac{z^n}{n!} = ze^{xz} = \sum_{n=1}^{\infty}\frac{x^{n-1}z^n}{(n-1)!}.$$

Hence we have

$$B_0(x+1) = B_0(x), \quad B_1(x+1) - B_1(x) = 1,$$

and in general

(6) $$B_n(x+1) - B_n(x) = nx^{n-1}, \qquad n = 2, 3, 4, \ldots,$$

from which it follows that

(7) $$B_n(1) = B_n(0) = B_n. \qquad n \geq 2$$

Since we have

$$\sum_{n=0}^{\infty} B_n(x+1)\frac{z^n}{n!} = \frac{ze^{zx}\, e^z}{e^z - 1} = \sum_{r=0}^{\infty} B_r(x)\frac{z^r}{r!}\sum_{m=0}^{\infty}\frac{z^m}{m!},$$

Cauchy's rule for multiplying power series gives a recurrence formula for the Bernoulli polynomial:

(8) $\sum_{r=0}^{n} \binom{n}{r} B_r(x) = B_n(x+1)$, or $\sum_{r=0}^{n-1} \binom{n}{r} B_r(x) = nx^{n-1}$

$n = 2, 3, 4, \ldots$.

From (5) and (6) we obtain

(9) $\int_x^y B_n(t)\,dt = \dfrac{B_{n+1}(y) - B_{n+1}(x)}{n+1}$, $\int_x^{x+1} B_n(t)\,dt = x^n.$

Hence it follows that

(10) $\sum_{r=0}^{m-1} r^n = \sum_{r=0}^{m-1} \int_r^{r+1} B_n(t)\,dt = \int_0^m B_n(t)\,dt = \dfrac{B_{n+1}(m) - B_{n+1}}{n+1}$.

$n = 2, 3, 4, \ldots$.

From (6) we can obtain the multiplication theorem and the symmetry property of $B_n(x)$ (Fort pp. 32, 34)

(11) $B_n(mx) = m^{n-1} \sum_{r=0}^{m-1} B_n(x + r/m)$,

(12) $B_n(1-x) = (-1)^n B_n(x)$.

The Bernoulli polynomials are expressible in trigonometric series. For $B_1(x)$ we have from (3)

(13) $B_1(x) = x - \tfrac{1}{2} = -\sum_{r=1}^{\infty} (r\pi)^{-1} \sin(2\pi r x)$ $\qquad 0 < x < 1.$

The Fourier series of $B_k(x)$ for $k > 1$ can easily be obtained by the calculus of residues. Consider $\int_C f(z)\,dz$ with $f(z) = z^{-k} e^{xz}(e^z - 1)^{-1}$ (k an integer > 1), the contour C being a (large) circle with radius $(2N+1)\pi$ (N an integer), center at the origin. The poles of the integrand are $z_r = 2\pi i r$, ($r = 0, \pm 1, \pm 2, \ldots$). The residues of the function $f(z)$ for $r = \pm 1, \pm 2, \ldots$ are easily found to be $(2\pi i r)^{-k} e^{2\pi i r x}$, and from (2) the residue at $z = 0$ is seen to be $B_k(x)/k!$.The integral around the circle C tends to zero as $N \to \infty$ provided $0 \le x \le 1$, and by the theorem of residues we have

$$B_k(x)/k! = -\sum_{r=-\infty}^{\infty}{}' (2\pi i r)^{-k} e^{2\pi i r x}.$$

The prime indicates that the term corresponding to $r = 0$ must be omitted. This gives the expansions ($n = 1, 2, 3, \ldots$; $0 \le x \le 1$)

(14) $B_{2n}(x) = 2(-1)^{n+1}(2n)! \sum_{r=1}^{\infty} (2\pi r)^{-2n} \cos(2\pi r x)$,

(15) $B_{2n+1}(x) = 2(-1)^{n+1}(2n+1)!\sum_{r=1}^{\infty}(2\pi r)^{-2n-1}\sin(2\pi rx).$

Putting $x = 0$ we get the following expressions for the Bernoulli numbers (cf. also Schwatt, 1932, p. 143):

(16) $B_{2n} = 2(-1)^{n+1}(2n)!\sum_{r=1}^{\infty}(2\pi r)^{-2n}$ $\qquad n = 1, 2, 3, \ldots,$

(17) $B_{2n+1} = 0$ $\qquad n = 1, 2, 3, \ldots.$

Equations (4) and (8) give a recurrence formula for the Bernoulli numbers

(18) $\sum_{r=0}^{n-1}\binom{n}{r}B_r = 0$ $\qquad n = 2, 3, 4, \ldots.$

From (18) and (3) we have

(19) $B_0 = 1, \quad B_1 = -\tfrac{1}{2}, \quad B_2 = 1/6, \quad B_4 = -1/30, \quad B_6 = 1/42, \ldots,$

Numerical values of the B_{2n} up to B_{40} and recurrence relations can be found in Ramanujan, 1927, p. 1.

By using (14), (15), and 1.11(4), the following integral representations for the Bernoulli polynomials are obtained:

(20) $B_{2n}(x) = (-1)^{n+1}(2n)\int_0^{\infty}\frac{\cos(2\pi x) - e^{-2\pi t}}{\cosh(2\pi t) - \cos(2\pi x)}t^{2n-1}\,dt$

$0 < \operatorname{Re} x < 1, \quad n = 1, 2, 3, \ldots,$

(21) $B_{2n+1}(x) = (-1)^{n+1}(2n+1)\int_0^{\infty}\frac{\sin(2\pi x)}{\cosh(2\pi t) - \cos(2\pi x)}t^{2n}\,dt$

$0 < \operatorname{Re} x < 1, \quad n = 0, 1, 2, \ldots.$

In terms of Riemann's zeta-function 1.12(1) we have

(22) $B_{2n} = (-1)^{n+1}(2\pi)^{-2n}\,2(2n)!\,\zeta(2n)$ $\qquad n = 0, 1, 2, \ldots,$

(23) $B_{2n} = -2n\,\zeta[-(2n-1)]$ $\qquad n = 1, 2, 3, \ldots,$

as is seen from (16) and 1.12(22).

From 1.12(4) to 1.12(8) we find integral representations for the B_{2n} $(n = 1, 2, 3, \ldots)$:

(24) $B_{2n} = (-1)^{n+1}\,4n\int_0^{\infty}t^{2n-1}(e^{2\pi t} - 1)^{-1}\,dt$

$= (-1)^{n+1}\,2n\int_0^{\infty}t^{2n-1}e^{-\pi t}\operatorname{csch}(\pi t)\,dt,$

(25) $B_{2n} = (-1)^{n+1}\, 4n(1-2^{1-2n})^{-1} \int_0^\infty t^{2n-1}\,(e^{2\pi t}+1)^{-1}\, dt$

$$= (-1)^{n+1}\, 2n\,(1-2^{1-2n})^{-1} \int_0^\infty t^{2n-1}\, e^{-\pi t}\operatorname{sech}(\pi t)\, dt,$$

(26) $B_{2n} = (-1)^{n+1}\, 2n\,(2^{2n}-1)^{-1} \int_0^\infty t^{2n-1} \operatorname{csch}(\pi t)\, dt,$

(27) $B_{2n} = (-1)^{n+1}\, \pi \int_0^\infty t^{2n}[\operatorname{csch}(\pi t)]^2\, dt,$

(28) $B_{2n} = (-1)^{n+1}\, \pi(1-2^{1-2n})^{-1} \int_0^\infty t^{2n}\,[\operatorname{sech}(\pi t)]^2 dt.$

(For other results cf. Nielsen, 1923, and Ramanujan, 1927, p.1.)

1.13.1. The Bernoulli polynomials of higher order

The Bernoulli numbers and polynomials of order m are defined respectively by

(29) $a_1 \cdots a_m\; z^m\,[(e^{a_1 z}-1)\cdots(e^{a_m z}-1)]^{-1}$

$$= \sum_{n=0}^{\infty} B_n^{(m)}(a_1 \cdots a_m)\; z^n/n! \qquad |z| < 2\pi|a_l|^{-1},$$

(30) $a_1 \cdots a_m\, z^m[(e^{a_1 z}-1)\cdots(e^{a_m z}-1)]^{-1}\, e^{xz}$

$$= \sum_{n=0}^{\infty} B_n^{(m)}(x|a_1 \cdots a_m)\; z^n/n! \qquad |z| < 2\pi|a_l|^{-1}.$$

Here m is a positive integer, $a_1, \ldots, a_m$ are arbitrary parameters, and

(31) $|a_l| = \max\,[\,|a_1|, \ldots, |a_m|\,].$

For $m = 1$ and $a_1 = 1$, (29) and (30) reduce to (1) and (2) respectively.

Clearly we have

(32) $B_n^{(m)}(0|a_1 \cdots a_m) = B_n^{(m)}(a_1 \cdots a_m),$

(33) $B_n^{(1)}(x|a_1) = a_1^n\, B_n(x/a_1).$

From (29) and (30)

(34) $B_n^{(m)}(x|a_1 \cdots a_m) = \sum_{l=0}^{n} \binom{n}{l} x^l\, B_{n-l}^{(m)}(a_1 \cdots a_m).$

We denote

(35) $\xi = \tfrac{1}{2}(a_1 + \cdots + a_m)$

and

(36) $D_n^{(m)} = 2^n\, B_n^{(m)}(\xi|a_1 \cdots a_m).$

It can be shown that

(37) $D_{2n+1}^{(m)} = 0$ $\qquad n = 0, 1, 2, \ldots .$

We thus get from (30)

(38) $(a_1 \cdots a_m) z^m [\sinh(a_1 z) \cdots \sinh(a_m z)]^{-1} = \sum_{n=0}^{\infty} D_{2n}^{(m)} z^{2n}/(2n)!$

$|z| < \pi |a_l|^{-1}.$

The Bernoulli numbers and polynomials of order $-m$ ($m = 1, 2, 3, \ldots$) are defined respectively by

(39) $(e^{a_1 z} - 1) \cdots (e^{a_m z} - 1)(a_1 \cdots a_m)^{-1} z^{-m}$

$= \sum_{n=0}^{\infty} B_n^{(-m)}(a_1 \cdots a_m)\, z^n/n!,$

(40) $(e^{a_1 z} - 1) \cdots (e^{a_m z} - 1)(a_1 \cdots a_m)^{-1} z^{-m} e^{xz}$

$= \sum_{n=0}^{\infty} B_n^{(-m)}(x|a_1 \cdots a_m)\, z^n/n!\,;$

both expansions converge in the whole z-plane.

From (35) and (40) for $x = -\xi$ we have

(41) $\sinh(a_1 z) \cdots \sinh(a_m z)(a_1 \cdots a_m)^{-1} z^{-m} = \sum_{n=0}^{\infty} D_{2n}^{(-m)} z^{2n}/(2n)!,$

where

(42) $D_n^{(-m)} = 2^n B_n^{(-m)}(-\xi|a_1 \cdots a_m).$

Again we have

(43) $D_{2n+1}^{(-m)} = 0$ $\qquad n = 0, 1, 2, \ldots .$

For an exhaustive treatise of the Bernoulli numbers and polynomials of higher order see Nörlund, 1922 and 1924, Ch. VI.

The case $a_1 = a_2 = \cdots = a_m = 1$ is thoroughly discussed in Milne-Thomson, 1933, Ch. VI.

1.14. Euler numbers and polynomials

Euler numbers E_n and Euler polynomials $E_n(x)$ are defined by the equations:

(1) $\operatorname{sech} z = 2e^z(e^{2z} + 1)^{-1} = \sum_{n=0}^{\infty} E_n z^n/n!$ $\qquad |z| < \tfrac{1}{2}\pi,$

(2) $2e^{xz}(e^z + 1)^{-1} = \sum_{n=0}^{\infty} E_n(x) z^n/n!$ $\qquad |z| < \pi.$

Differentiating (2) with respect to x and equating coefficients of z^n we obtain

(3) $E_n'(x) = n\, E_{n-1}(x).$

If the left-hand side of (2) is written in the form

see errata!

$$2e^{z/2}(e^z+1)^{-1}\, e^{z(x-\frac{1}{2})} = \sum_{r=0}^{\infty} E_r z^r (r!\, 2^r)^{-1} \sum_{m=0}^{\infty} (x-\tfrac{1}{2})^m z^m/m!\,,$$

Cauchy's rule for multiplying power series gives

(4) $E_n(x) = \sum_{r=0}^{n} \binom{n}{r} 2^{-r} E_r (x-\tfrac{1}{2})^{n-r},$

hence taking $x = \tfrac{1}{2}$,

(5) $E_n = 2^n E_n(\tfrac{1}{2}).$

From (2) we have

$$\sum_{n=0}^{\infty} [E_n(x+1) + E_n(x)]\, z^n/n! = 2e^{xz} = 2\sum_{n=0}^{\infty} x^n z^n/n!\,,$$

and therefore

(6) $E_n(x+1) + E_n(x) = 2x^n.$

Writing

$$ze^{xz/2}(e^{z/2}+1)^{-1} = ze^{(x+1)z/2}(e^z-1)^{-1} - ze^{xz/2}(e^z-1)^{-1},$$

we obtain from (2) and 1.13(2)

(7) $E_{n-1}(x) = n^{-1}\, 2^n \{B_n[\tfrac{1}{2}(x+1)] - B_n(\tfrac{1}{2}x)\} = n^{-1}\, 2[B_n(x) - 2^n B_n(\tfrac{1}{2}x)].$

Hence from 1.13(11), 1.13(12) the following relations are obtained

(8) $E_n(mx) = m^n \sum_{r=0}^{m-1} (-1)^r E_n(x + r/m)$ $\qquad m$ odd,

(9) $E_n(mx) = -2m^n(n+1)^{-1} \sum_{r=0}^{m-1} (-1)^r B_{n+1}(x + r/m)$ $\qquad m$ even,

(10) $E_n(1-x) = (-1)^n E_n(x).$

From

$$2e^{(x+1)z}(e^z+1)^{-1} = \sum_{r=0}^{\infty} E_r(x)\, z^r/r! \sum_{m=0}^{\infty} z^m/m! = \sum_{n=0}^{\infty} E_n(x+1)\, z^n/n!$$

we obtain a recurrence formula

(11) $\sum_{r=0}^{n} \binom{n}{r} E_r(x) = E_n(x+1),$ or $\sum_{r=0}^{n} \binom{n}{r} E_r(x) + E_n(x) = 2x^n.$

In a manner similar as in 1.13 a representation of Euler's polynomials by means of Fourier series can be obtained. Here one considers the integral $2\int_c z^{-k-1} e^{xz} (e^z+1)^{-1}\,dz$ taken along a circle, center at the origin, radius $2N\pi$ (N an integer). From (2) the residue of the integrand at $z=0$ is easily seen to be $E_k(x)/k!$. The result is

$$(12)\quad E_{2n}(x) = (-1)^n 4(2n)! \sum_{r=0}^{\infty} [(2r+1)\pi]^{-2n-1} \sin[(2r+1)\pi x]$$
$$n = 1, 2, 3, \ldots, \quad 0 \le x \le 1,$$

$$(13)\quad E_{2n+1}(x) = (-1)^{n+1} 4(2n+1)! \sum_{r=0}^{\infty} [(2r+1)\pi]^{-2n-2} \cos[(2r+1)\pi x]$$
$$n = 0, 1, 2, \ldots, \quad 0 \le x \le 1.$$

From (5), (12), and (13) we have

$$(14)\quad E_{2n} = (-1)^n 2(2n)!\,(2/\pi)^{2n+1} \sum_{r=0}^{\infty} (-1)^r/(2r+1)^{2n+1} \qquad n = 0, 1, 2, \ldots,$$

$$(15)\quad E_{2n+1} = 0,$$

or, with the notation of 1.12(27)

$$(16)\quad E_{2n} = (-1)^n 2(2n)!\,(2/\pi)^{2n+1} L(2n+1) \qquad n = 0, 1, 2, \ldots.$$

The equation

$$(1/\cosh z)\cosh z = 1 = \sum_{n=0}^{\infty} E_{2n}\, z^{2n}/(2n)! \sum_{m=0}^{\infty} z^{2m}/(2m)!,$$

and the application of Cauchy's multiplication rule gives the recurrence formula for Euler's numbers:

$$(17)\quad \sum_{r=0}^{n} \binom{2n}{2r} E_{2r} = 0 \qquad n > 0.$$

Using (14) we have

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385, \ldots.$$

An integral expression for E_{2n} can be obtained by replacing $L(2n+1)$ in (16) by the expression 1.12(28),

$$(18)\quad E_{2n} = (-1)^n (2/\pi)^{2n+1} \int_0^{\infty} t^{2n} \operatorname{sech} t\, dt$$
$$= (-1)^n 2^{2n+1} \int_0^{\infty} t^{2n} \operatorname{sech}(\pi t)\, dt \qquad n = 0, 1, 2, \ldots.$$

The Fourier expansions (12) and (13) can be replaced by integral expressions. The result is:

$$(19)\quad E_{2n}(x) = (-1)^n\, 4 \int_0^\infty \frac{t^{2n} \sin(\pi x)\, \cosh(\pi t)}{\cosh(2\pi t) - \cos(2\pi x)} dt$$

$n = 0, 1, 2, \ldots, \quad 0 < \mathrm{Re}\, x < 1,$

$$(20)\quad E_{2n+1}(x) = (-1)^{n+1}\, 4 \int_0^\infty \frac{t^{2n+1} \cos(\pi x)\, \sinh(\pi t)}{\cosh(2\pi t) - \cos(2\pi x)} dt$$

$n = 0, 1, 2, \ldots, \quad 0 < \mathrm{Re}\, x < 1.$

(For other results cf. Nielsen, 1923.)

1.14.1. The Euler polynomials of higher order

Euler's numbers and polynomials are defined respectively by

$$(21)\quad 2^m e^{z(\alpha_1 + \ldots + \alpha_m)} [(e^{2\alpha_1 z} + 1) \cdots (e^{2\alpha_m z} + 1)]^{-1}$$

$$= [\cosh(\alpha_1 z) \cdots \cosh(\alpha_m z)]^{-1} = \sum_{n=0}^{\infty} E_n^{(m)}(\alpha_1 \cdots \alpha_m)\, z^n/n!\,,$$

$$(22)\quad 2^m e^{xz} [(e^{\alpha_1 z} + 1) \cdots (e^{\alpha_m z} + 1)]^{-1} = \sum_{n=0}^{\infty} E_n^{(m)}(x|\alpha_1 \cdots \alpha_m)\, z^n/n!\,.$$

The series in formula (21) is convergent for $|z| < \tfrac{1}{2}\pi |\alpha_l|^{-1}$, and the series in (22) is convergent for $|z| < \pi |\alpha_l|^{-1}$ where $|\alpha_l|$ is defined in 1.13(31). Again in (21) and (22) m is a positive integer, and $\alpha_1, \ldots, \alpha_m$ are arbitrary parameters. The special case $m = 1, \alpha_1 = 1$ reduces to that discussed in 1.14.

Clearly from (21), (22), and 1.13(35) we have

$$(23)\quad E_n^{(m)}(\alpha_1 \cdots \alpha_m) = 2^n E_n^{(m)}(\xi|\alpha_1 \cdots \alpha_m).$$

The Euler numbers and polynomials of order $-m$ ($m = 1, 2, 3, \ldots$) are defined respectively as follows:

$$(24)\quad 2^{-m} e^{-z(\alpha_1 + \ldots + \alpha_m)} [(e^{2\alpha_1 z} + 1) \cdots (e^{2\alpha_m z} + 1)]$$

$$= \cosh(\alpha_1 z) \cdots \cosh(\alpha_m z) = \sum_{n=0}^{\infty} E_n^{(-m)}(\alpha_1 \cdots \alpha_m)\, z^n/n!\,,$$

$$(25)\quad 2^{-m} e^{xz} (e^{\alpha_1 z} + 1) \cdots (e^{\alpha_m z} + 1) = \sum_{n=0}^{\infty} E_n^{(-m)}(x|\alpha_1 \cdots \alpha_m)\, z^n/n!\,;$$

both expansions are convergent in the whole z-plane. For more details see Nörlund, 1922 and 1924, Ch. VI. The case $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 1$ is thoroughly discussed in Milne-Thomson, 1933, Ch. VI.

1.15. Some integral formulas connected with the Bernoulli and Euler polynomials

Some integral relations can be deduced from the two preceding sections.

First, 1.13(1) can be written in the form

(1) $$(e^z-1)^{-1}-z^{-1}+\tfrac{1}{2}=\sum_{n=1}^{\infty} B_{2n}\, z^{2n-1}/(2n)! \qquad |z|<2\pi.$$

If the B_{2n} are replaced by 1.13(24) and 1.13(27) we find

(2) $$(e^z-1)^{-1}=z^{-1}-\tfrac{1}{2}+2\int_0^\infty (e^{2\pi t}-1)^{-1}\sin(tz)\,dt \qquad |\operatorname{Im} z|<2\pi,$$

(3) $$(e^{2z}-1)^{-1}=(2z)^{-1}-\tfrac{1}{2}+\pi z^{-1}\int_0^\infty \sin^2(tz)\operatorname{csch}^2(\pi t)\,dt \qquad |\operatorname{Im} z|<\pi.$$

If in 1.13(2) the $B_r(x)$ are replaced by the expressions 1.13(20) and 1.13(21) and in 1.14(2) the $E_r(x)$ by the expressions 1.14(19) and 1.14(20), we find

(4) $$\frac{e^{xz}}{e^z-1}=\frac{1}{z}+\int_0^\infty \frac{\cos(2\pi x)-e^{-2\pi t}}{\cosh(2\pi t)-\cos(2\pi x)}\sin(tz)\,dt$$
$$-\int_0^\infty \frac{\sin(2\pi x)}{\cosh(2\pi t)-\cos(2\pi x)}\cos(tz)\,dt$$
$$0\le x<1, \qquad |\operatorname{Im} z|<2\pi,$$

(5) $$\frac{e^{xz}}{e^z+1}=2\int_0^\infty \frac{\sin(\pi x)\cosh(\pi t)}{\cosh(2\pi t)-\cos(2\pi x)}\cos(tz)\,dt$$
$$-2\int_0^\infty \frac{\cos(\pi x)\sinh(\pi t)}{\cosh(2\pi t)-\cos(2\pi x)}\sin(tz)\,dt$$
$$0\le x<1, \qquad |\operatorname{Im} z|<\pi.$$

1.16. Polygamma functions

We define

(1) $$\psi^{(n)}(z)=\frac{d^{n+1}\log\Gamma(z)}{dz^{n+1}}=\frac{d^n\psi(z)}{dz^n}, \qquad \psi^{(0)}(z)=\psi(z)$$
$$n=1,2,3,\ldots,$$

(2) $$G^{(n)}(z)=\frac{d^n G(z)}{dz^n}, \qquad G^{(0)}(z)=G(z) \qquad n=1,2,3,\ldots.$$

The following functional equations are consequences of the results of 1.7.1 and 1.8:

(3) $$\psi^{(n)}(z)-\psi^{(n)}(1+z)=(-1)^{n+1}\,n!/z^{n+1}$$

(4) $$\psi^{(n)}(z)-(-1)^n\psi^{(n)}(1-z)=-\pi\frac{d^n}{dz^n}[\operatorname{ctn}(\pi z)]$$

(5) $$\psi^{(n)}(mz)=m^{-n-1}\sum_{r=0}^{m-1}\psi^{(n)}(z+r/m) \qquad m=1,2,3,\ldots,$$

(6) $2^n G^{(n)}(z) = \psi^{(n)}(\tfrac{1}{2}z + \tfrac{1}{2}) - \psi^{(n)}(\tfrac{1}{2}z)$,

(7) $G^{(n)}(1+z) + G^{(n)}(z) = 2(-1)^n n!/z^{n+1}$,

(8) $G^{(n)}(z) + (-1)^n G^{(n)}(1-z) = 2\pi \dfrac{d^n}{dz^n}\,[\csc(\pi z)]$.

We have also the expressions:

(9) $\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{r=0}^{\infty} (z+r)^{-n-1} = (-1)^{n+1} n!\ \zeta(n+1, z)$,

(10) $G^{(n)}(z) = 2(-1)^n n! \sum_{r=0}^{\infty} (-1)^r (z+r)^{-n-1} = 2(-1)^n n!\, \Phi(-1, n+1, z)$.

Hence, we may express $\psi^{(n)}(z)$ and $G^{(n)}(z)$ as definite integrals if we replace the functions ζ and Φ by their integral representations.

1.17. Some expansions for $\log\Gamma(1+z)$, $\psi(1+z)$, $G(1+z)$, and $\Gamma(z)$

The Taylor expansion of $\log\Gamma(1+z)$ is

(1) $\log\Gamma(1+z) = \sum_{m=0}^{\infty} \left[\dfrac{d^m \log\Gamma(1+z)}{dz^m}\right]_{z=0} \dfrac{z^m}{m!}$

$$= z\,\psi(1) + \sum_{m=2}^{\infty} z^m/m!\,[\psi^{(m-1)}(1+z)]_{z=0}$$

or

(2) $\log\Gamma(1+z) = -\gamma z + \sum_{m=2}^{\infty} (-1)^m \zeta(m)\, z^m/m \qquad |z| < 1$,

[cf. 1.16(9) and 1.12(1)].

Taking $z = 1$ we obtain the expression

(3) $\gamma = \sum_{m=2}^{\infty} (-1)^m \zeta(m)/m$

for Euler's constant.

If in

(4) $\psi(1+z) = -\gamma + \sum_{n=1}^{\infty} \left(\dfrac{1}{n} - \dfrac{1}{z+n}\right)$

[cf. 1.7(3)] we expand

$$\frac{1}{n} - \frac{1}{z+n} = \frac{z}{n^2} - \frac{z^2}{n^3} + \frac{z^3}{n^4} - \cdots \qquad |z| < 1,$$

we obtain

(5) $\psi(1+z) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n)\, z^{n-1} \qquad |z| < 1.$

Similarly, from 1.8(6) we have

(6) $$G(1+z) = 2\sum_{n=1}^{\infty} (-1)^{n-1}\,\sigma(n)\,z^{n-1}$$

$$= 2\sigma(1) + 2\sum_{n=2}^{\infty} (-1)^{n-1}(1-2^{1-n})\,\zeta(n)\,z^{n-1}$$

where $|z| < 1$, $\sigma(1) = \log 2$, and $\sigma(n) = \sum_{r=1}^{\infty} (-1)^{r-1}/r^n = (1-2^{1-n})\,\zeta(n)$ for $n > 1$

If we form the expressions $\psi(1+z) + \psi(1-z)$ and $G(1+z) + G(1-z)$ by means of (5) and (6), and take into account 1.8(7), 1.8(8), 1.7(10), and 1.7(11) we obtain

(7) $$\psi(1+z) = (2z)^{-1} - \gamma - (\pi/2)\,\mathrm{ctn}(\pi z) - \sum_{n=1}^{\infty} \zeta(2n+1)\,z^{2n} \qquad |z| < 1,$$

(8) $$G(1+z) = z^{-1} - \pi\csc(\pi z) + 2\sigma(1) + 2\sum_{n=1}^{\infty}(1-2^{-2n})\,\zeta(2n+1)\,z^{2n} \qquad |z| < 1.$$

Using 1.7(1) we have from (7)

(9) $$\log\Gamma(1+z) = -\tfrac{1}{2}\log\left(\frac{\sin \pi z}{\pi z}\right) - \sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{2n+1}\,z^{2n+1} - \gamma z,$$

or, using the series

$$\tfrac{1}{2}\log\left(\frac{1+z}{1-z}\right) = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1},$$

we obtain

(10) $$\log\Gamma(1+z) = \tfrac{1}{2}\left\{\log\left[\frac{\pi z}{\sin(\pi z)}\right] - \log\left(\frac{1+z}{1-z}\right)\right\} + \sum_{n=1}^{\infty}\frac{1-\zeta(2n+1)}{2n+1}\,z^{2n+1} + (1-\gamma)\,z.$$

Formulas (9) and (10) are valid if $|z| < 1$.

Finally we give an expression of $\Gamma(z)$ and $\psi(z)$ near $z = -m$ ($m = 0, 1, 2, \ldots$). From 1.2(6) we have

$$\Gamma(z) = \pi(-1)^m/\{\Gamma(1-z)\sin[\pi(z+m)]\}.$$

Expanding $1/\Gamma(1-z)$ in a Taylor series near $z = -m$ and using 1.13(36) we obtain

(11) $$\Gamma(z) = [(-1)^m/m!]\{(z+m)^{-1} + \psi(m+1) + \tfrac{1}{2}(z+m)[(\pi^2/3) + \psi^2(m+1) - \psi'(m+1)] + O[(z+m)^2]\}.$$

Similarly from 1.7(11), 1.13(31), and 1.16(9) we have

(12) $\psi(z) = -(z+m)^{-1} + \psi(m+1) + \sum_{n=2}^{\infty} [(-1)^n \zeta(n) + \sum_{r=1}^{m} r^{-n}] (z+m)^{n-1}.$

1.18. Asymptotic expansions

In 1.9(5) we replace the expression within the parentheses under the integral sign by the right-hand side of 1.15(1). Since the conditions of Watson's lemma are satisfied, we may integrate term by term and obtain the following asymptotic expansion (Stirling series)

(1) $$\log\Gamma(z) = (z-\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log(2\pi) + \sum_{n=1}^{m} B_{2n}/[(2n-1)(2n)z^{2n-1}] + O(z^{-2m-1}) \qquad |\arg z| < \pi.$$

This is equivalent to

(2) $$\Gamma(z) = e^{-z} e^{(z-\frac{1}{2})\log z} (2\pi)^{\frac{1}{2}} \left[1 + \frac{z^{-1}}{12} + \frac{z^{-2}}{288} - \frac{139\, z^{-3}}{51840} - O(z^{-4})\right] \qquad |\arg z| < \pi.$$

[Formula (2) can be obtained directly from the loop integral 1.6(2) using the method of steepest descent. For this and for the remainder in (1) and (2) cf. Watson, 1920, p. 1.]

From (1) and (2) a number of asymptotic formulas can be derived, such as

(3) $\log\Gamma(z+a) = (z+a-\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log(2\pi) + O(z^{-1}),$

(4) $\Gamma(z+a)/\Gamma(z+\beta) = z^{a-\beta}[1 + \tfrac{1}{2}z^{-1}(a-\beta)(a+\beta-1) + O(z^{-2})],$

(5) $\lim_{|z|\to\infty} e^{-a\log z}\,\Gamma(z+a)/\Gamma(z) = 1.$

In connection with formula (3) see also (12), and in connection with (4) see (13). In (3), (4), and (5) a and β are fixed arbitrary complex numbers and $-\pi < \arg z < \pi$. We also have

(6) $\lim_{|y|\to\infty} |\Gamma(x+iy)|\, e^{\frac{1}{2}\pi|y|}\, |y|^{\frac{1}{2}-x} = (2\pi)^{\frac{1}{2}}$ $\qquad x, y$ real.

From (1) we obtain the asymptotic expansion for $\psi(z)$,

(7) $\psi(z) = \log z - (2z)^{-1} - \sum_{n=1}^{m} B_{2n} z^{-2n}/(2n) + O(z^{-2m-2}).$

The integrand of 1.10(4) can be written as [cf. 1.15(1)]

(8) $t^{s-1} e^{-vt}/(1-e^{-t}) = t^{s-1} e^{-vt}[t^{-1} + \tfrac{1}{2} + \sum_{n=1}^{\infty} B_{2n} t^{2n-1}/(2n)!]\,.$

Hence, from 1.10(3) we obtain the following asymptotic expansion of $\zeta(s, v)$ for large values of $|v|$ with $|\arg v| < \pi$:

(9) $$\zeta(s, v) = [1/\Gamma(s)]\{v^{1-s}\Gamma(s-1) + \tfrac{1}{2}v^{-s}\Gamma(s) + \sum_{n=1}^{m} B_{2n}\Gamma(s+2n-1)/[(2n)!\,v^{2n+s-1}] + O(v^{-2m-s-1})\}$$
Re $s > 1$.

Putting $s = (n+1)$ we obtain an asymptotic expansion for $\psi^{(n)}(z)$ as given in 1.16(9).

Finally we derive an asymptotic expression of $\log\Gamma(z)$ due to Binet. In Binet's first expression 1.9(4) we write the integrand in the form

$$\tfrac{1}{2}e^{-tz}t^{-2}(e^t-1)^{-1}[e^t(t-2)+t+2]$$
$$= \tfrac{1}{2}\sum_{n=1}^{\infty} nt^n e^{-tz}/[(n+2)!\,(e^t-1)],$$

replacing e^t in the numerator on the left-hand side by its power series. Since, according to 1.10(3)

$$\int_0^\infty t^n e^{-tz}(e^t-1)^{-1}\,dt = \Gamma(n+1)\,\zeta(n+1,\ z+1),$$

we obtain

(10) $$\log\Gamma(z) = (z-\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log(2\pi) + \tfrac{1}{2}\sum_{n=1}^{\infty}\frac{n}{(n+1)(n+2)}\zeta(n+1,\ z+1).$$

This is Binet's formula.

A similar expression converging faster is Burnside's formula (Wilton, 1922, p. 90)

(11) $$\log\Gamma(z) = (z-\tfrac{1}{2})\log(z-\tfrac{1}{2}) - z - \tfrac{1}{2} + \tfrac{1}{2}\log(2\pi) - \sum_{n=1}^{\infty}\zeta(2n,\ z)/[2^{2n}\,2n\,(2n+1)]$$
Re $z \geq -\tfrac{1}{2}$.

From the left-hand side of (3) and (4) complete asymptotic expansions can be given. These are

(12) $$\log\Gamma(z+a) = (z+a-\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log(2\pi) + \frac{B_2(a)\,z^{-1}}{1\cdot 2} - \cdots + \frac{(-1)^{n+1}B_{n+1}(a)\,z^{-n}}{n\,(n+1)} + O(z^{-n-1})$$
$|\arg z| < \pi$, $n = 1, 2, 3, \ldots$.

(13) $\Gamma(z+\alpha_1)\,\Gamma(z+\alpha_2)/[\Gamma(z+\beta_1)\,\Gamma(z+\beta_2)]$

$$= z^{\alpha_1+\alpha_2-\beta_1-\beta_2}\left[1+\frac{c_1}{z+1}+\frac{c_2}{(z+1)(z+2)}+\cdots\right] \qquad |\arg z|<\pi.$$

These expansions are due to Barnes, 1899, p. 64, and Van Engen, 1938, respectively.

1.19. Mellin-Barnes integrals

Of all the integrals which contain gamma functions in their integrands the most important ones are the so-called Mellin-Barnes integrals. Such integrals were first introduced by S. Pincherle, in 1888; their theory has been developed by H. Mellin (1910, where there are references to earlier work), and they were used for a complete integration of the hypergeometric differential equation by E. W. Barnes (1908). See also section 2.1.3.

The integral

(1) $$f(z)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(a_1+A_1s)\cdots\Gamma(a_m+A_ms)}{\Gamma(c_1+C_1s)\cdots\Gamma(c_p+C_ps)}\times\frac{\Gamma(b_1-B_1s)\cdots\Gamma(b_n-B_ns)}{\Gamma(d_1-D_1s)\cdots\Gamma(d_q-D_qs)}\,z^s\,ds$$

is a typical Mellin-Barnes integral. It will be assumed that γ is real, all the A_j, B_j, C_j, D_j are positive, and that the path of integration is a straight line parallel to the imaginary axis with indentations, if necessary, to avoid the poles of the integrand. The discussion given here is based on Dixon and Ferrar (1936).

The following notations will be used:

(2) $$\alpha=\sum_{j=1}^{m}A_j+\sum_{j=1}^{n}B_j-\sum_{j=1}^{p}C_j-\sum_{j=1}^{q}D_j$$

(3) $$\beta=\sum_{j=1}^{m}A_j-\sum_{j=1}^{n}B_j-\sum_{j=1}^{p}C_j+\sum_{j=1}^{q}D_j$$

(4) $$\lambda=\mathrm{Re}\,\Big(\sum_{j=1}^{m}a_j-\tfrac12 m+\sum_{j=1}^{n}b_j-\tfrac12 n-\sum_{j=1}^{p}c_j+\tfrac12 p-\sum_{j=1}^{q}d_j+\tfrac12 q\Big)$$

(5) $$\rho=\prod_{j=1}^{m}(A_j)^{A_j}\prod_{j=1}^{n}(B_j)^{-B_j}\prod_{j=1}^{p}(C_j)^{-C_j}\prod_{j=1}^{q}(D_j)^{D_j}.$$

The convergence of (1) can be investigated by means of the asymptotic representation of the gamma function 1.18(6). With

$$s=\sigma+it\quad(\sigma,\,t\ \text{real}),\qquad z=R\,e^{i\Phi}\quad(R>0,\quad\Phi\ \text{real})$$

the absolute value of the integrand is comparable with

(6) $$e^{-\frac{1}{2}\alpha\pi|t|}\,|t|^{\beta\gamma+\lambda}R^{-\gamma}\,e^{\Phi t}\rho^{\gamma}$$

when $|t|$ is large. There are four types of convergent integrals (1).

First type: $\alpha > 0$. The integral converges absolutely for $|\Phi| < \alpha\pi/2$ and defines a function analytic in the sector $|\arg z| < \min(\pi, \alpha\pi/2)$. (The point $z = 0$ is tacitly excluded.)

Second type: $\alpha = 0$, $\beta \neq 0$. The integral (1) does not converge for complex z. For $z > 0$ it converges absolutely if γ is so chosen that

(7) $$-\beta\gamma > 1 + \lambda;$$

and there exists an analytic function of z, defined over $|\arg z| < \pi$, whose values for positive z are given by (1).

Third type: $\alpha = \beta = 0$, $\lambda < -1$. Here (7) is satisfied for arbitrary γ. The integral converges absolutely for all positive z (but not for complex z) and represents a continuous function of z $(0 < z < \infty)$. There are nowtwo analytic functions, one regular in any domain contained in $|\arg z| < \pi$, $|z| > \rho$ whose values for $z > \rho$ are represented by (1), and another regular in any domain contained in $|\arg z| < \pi$, $0 < |z| < \rho$ whose values for $0 < z < \rho$ are represented by (1). The two functions are in general distinct.

Fourth type: $\alpha = \beta = 0, -1 \leq \lambda < 0$. The integral converges (although not absolutely) for $0 < z < \rho$ and for $z > \rho$. There are two analytic functions of the same nature as in the preceding case. There is a discontinuity at $z = \rho$ and the integral does not exist there, though it may have a principal value. The nature of the discontinuity, and the principal value, are discussed in the paper by Dixon and Ferrar.

Multiple integrals of a similar structure occur occasionally.

An example for an integral of the Mellin-Barnes type is the following one (Whittaker-Watson, 1927, p. 289)

(8) $$\int_{i\infty}^{i\infty}\Gamma(\alpha+s)\,\Gamma(\beta+s)\,\Gamma(\gamma-s)\,\Gamma(\delta-s)\,ds = 2\pi i\,\frac{\Gamma(\alpha+\gamma)\,\Gamma(\alpha+\delta)\,\Gamma(\beta+\gamma)\,\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}.$$

The path of integration is indented so that the poles of $\Gamma(\gamma - s)\,\Gamma(\delta - s)$ lie to the right and the poles of $\Gamma(\alpha + s)\,\Gamma(\beta + s)$ to the left of it, and itis supposed that α, β, γ, δ are such that no pole of the first set coincides with any pole of the second set. [For further examples cf. 2.1(15) and section 7.3.6 and Ramanujan, 1927, p. 216.]

1.20. Power series of some trigonometric functions

From 1.13(1) a number of trigonometric expansions can be deduced (cf. also similar expansions obtained by Schwatt, 1932) such as

(1) $$z \coth z = 2z(e^{2z}-1)^{-1} + z = \sum_{n=0}^{\infty} 2^{2n} B_{2n}\, z^{2n}/(2n)!$$

$$= 2\sum_{n=0}^{\infty} (-1)^{n+1} \zeta(2n)\, \pi^{-2n} z^{2n} \qquad |z| < \pi,$$

(2) $$\tanh z = 2\coth(2z) - \coth z = \sum_{n=1}^{\infty} 2^{2n}(2^{2n}-1) B_{2n}\, z^{2n-1}/(2n)!$$

$$= 2\sum_{n=1}^{\infty} (-1)^{n+1} (2^{2n}-1)\, \zeta(2n)\, \pi^{-2n} z^{2n-1} \qquad |z| < \pi/2,$$

(3) $$z \operatorname{ctn} z = \sum_{n=0}^{\infty} (-1)^n 2^{2n} B_{2n}\, z^{2n}/(2n)!$$

$$= -2\sum_{n=0}^{\infty} \zeta(2n)\, \pi^{-2n} z^{2n} \qquad |z| < \pi,$$

(4) $$\tan z = \sum_{n=1}^{\infty} (-1)^{n+1} 2^{2n}(2^{2n}-1) B_{2n}\, z^{2n-1}/(2n)!$$

$$= 2\sum_{n=1}^{\infty} (2^{2n}-1)\, \zeta(2n)\, \pi^{-2n} z^{2n-1} \qquad |z| < \pi/2,$$

(5) $$z/\sin z = z\,[\operatorname{ctn}(\tfrac{1}{2}z) - \operatorname{ctn} z] = 2\sum_{n=0}^{\infty} (-1)^n (1-2^{2n-1}) B_{2n}\, z^{2n}/(2n)! \qquad |z| < \pi,$$

(6) $$\log\cos z = -\int_0^z \tan z\, dz = \sum_{n=1}^{\infty} (-1)^n (2^{2n}-1)\, 2^{2n-1} B_{2n}\, z^{2n}/[n(2n)!] \qquad |z| < \pi/2.$$

We write

(7) $$\tan z = \sum_{n=0}^{\infty} (-1)^{n+1} C_{2n+1}\, z^{2n+1}/(2n+1)! \qquad |z| < \pi/2,$$

(8) $$z/\sin z = \sum_{n=0}^{\infty} (-1)^n D_{2n}\, z^{2n}/(2n)! \qquad |z| < \pi.$$

Then a comparison with (4) and (5) gives

(9) $$C_{2n-1} = 2^{2n}(1-2^{2n})\, B_{2n}/(2n),$$

(10) $$D_{2n} = 2(1-2^{2n-1})\, B_{2n}\, .$$

Integral expressions for C_{2n-1} and D_{2n} can be obtained from 1.13(24) to 1.13(28).

More general expansions than those listed before can be obtained from the results in sections 1.13.1 and 1.14.1 (Nörlund, 1922, p. 196). Two examples are

(11) $\cos(mt)\,(t/\sin t)^m = \sum_{n=0}^{\infty} (-1)^n (2t)^{2n} B_{2n}^{(m)}/(2n)!\,,$

(12) $\sin(mt)\,(t/\sin t)^m = \sum_{n=0}^{\infty} (-1)^{n+1} (2t)^{2n+1} B_{2n+1}^{(m)}/(2n+1)!\,.$

Both expansions converge for $|t| < \pi$. With the notation used in 1.13(1) we have

(13) $B_l^{(m)} = B_l^{(m)}(a_1 \cdots a_m) \qquad a_1 = \cdots = a_m = 1.$

1.21. Some other notations and symbols

Alternative notations for the gamma function and some related symbols are (cf. 1.2):

(1) (Factorial function) $\Pi(z) = z! = \Gamma(z+1)$;

(2) γ = Euler's constant 1.1(4);

(3) (Hankel's symbol)

$$(v,n) = 2^{-2n}\{(4v^2-1)(4v^2-3^2)\cdots[4v^2-(2n-1)^2]\}/n!$$
$$= \Gamma(\tfrac12+v+n)/[n!\,\Gamma(\tfrac12+v-n)] \qquad n = 1, 2, 3, \ldots;$$

(4) (Kramp's symbol)

see errata!

$$c^{a/b} = c(c+b)(c+2b)\cdots[c+(a-1)b]$$
$$= b^{a}\,\Gamma(a+c/b)/\Gamma(c/b) \qquad a = 2, 3, 4, \ldots;$$

(5) (Pochhammer's symbol)

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$$
$$n = 1, 2, 3, \ldots;$$

(6) (Binomial coefficient)

$$\binom{a}{m} = (-1)^m\,\Gamma(m-a)/[m!\,\Gamma(-a)] = \Gamma(1+a)/[m!\,\Gamma(1+a-m)].$$

The Bernoulli numbers B_n are often defined by the expansion

(7) $\tfrac12 z + z(e^z-1)^{-1} = \tfrac12 z\coth(\tfrac12 z) = 1 - \sum_{n=1}^{\infty} (-1)^n B_n z^{2n}/(2n)!\,.$

It follows from 1.13(1) and 1.13(16) for the B_n thus defined

(8) $B_n = 2(2n)!\,(2\pi)^{-2n} \sum_{r=1}^{\infty} r^{-2n}$

and hence

(9) $B_1 = 1/6, \quad B_2 = 1/30, \quad B_3 = 1/42, \quad B_4 = 1/30, \ldots .$

The Bernoulli polynomials are often denoted by $\Phi_n(x)$ and defined by

(10) $z(e^{xz} - 1)/(e^z - 1) = \sum_{n=1}^{\infty} \Phi_n(x)\, z^n/n!\,.$

With our notation 1.13 (2) we have

(11) $\Phi_n(x) = B_n(x) - B_n(0)$

and hence with 1.13 (3)

(12) $\Phi_1(x) = x, \quad \Phi_2(x) = x^2 - x, \quad \Phi_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$

If the Euler numbers E_n are defined by

(13) $\operatorname{sech} z = \sum_{n=0}^{\infty} (-1)^n E_n\, z^{2n}/(2n)!\,,$

then it is obvious from 1.14 (1) and 1.14 (14) that

(14) $E_n = 2(2n)!\,(2/\pi)^{2n+1} \sum_{r=0}^{\infty} (-1)^r (2r+1)^{-2n-1}$

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CHAPTER II

THE HYPERGEOMETRIC FUNCTION

FIRST PART: THEORY

2.1. The hypergeometric series

2.1.1. The hypergeometric equation

If a homogeneous linear differential equation of the second order has at most three singularities we may assume that these are at 0, ∞, 1. If all of these singularities are "regular" (cf. Poole, 1936), then the equation can be reduced to the form (cf. Poole, 1936)

$$(1) \quad z(1-z)\frac{d^2u}{dz^2} + [c-(a+b+1)z]\frac{du}{dz} - abu = 0$$

where a, b, c, are independent of z. This is the *hypergeometric equation*. We shall call a, b, c the parameters of the equation; they are arbitrary complex numbers.

We define

$$(a)_n = \Gamma(a+n)/\Gamma(a),$$

i.e.,

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) \qquad n = 1, 2, 3, \ldots .$$

If $c \neq 0, -1, -2, \ldots$, then

$$(2) \quad u_1 = \sum_{n=0}^{\infty} (a)_n (b)_n z^n/[(c)_n n!] \equiv {}_2F_1(a, b; c; z) \equiv F(a, b; c; z)$$

is a solution of (1) which is regular at $z = 0$.

If $c = -n$, where $n = 0, 1, 2, \ldots$, then

$$(3) \quad u_1 = z^{n+1} \sum_{m=0}^{\infty} (a+n+1)_m (b+n+1)_m z^m/[(n+2)_m m!]$$

$$= z^{n+1} {}_2F_1(a+n+1, b+n+1; n+2; z)$$

is such a solution. The function ${}_2F_1(a, b; c; z)$ is called the hypergeometric series of variable z with parameters a, b, c. The subscripts in ${}_2F_1$ are usually omitted if there do not occur any other types of generalized hypergeometric series (cf. Chapters 4, 5) in the investigation.

We shall supplement the definition of the hypergeometric series in the case $c = -m$, $(m = 0, 1, 2, \ldots,)$, when (2) becomes meaningless.

If $a = -n$ or $b = -n$ where $n = 0, 1, 2, \ldots$, and if $c = -m$ where $m = n, n+1, n+2, \ldots$, then we define

$$(4)\quad \begin{cases} F(-n, b; -m; z) = \sum_{r=0}^{n} (-n)_r (b)_r z^r / [(-m)_r \, r!] \\ F(a, -n; -m; z) = \sum_{r=0}^{n} (a)_r (-n)_r z^r / [(-m)_r \, n!] \end{cases}$$

see errata!

Since (3) and (4) are solutions of (1), we see that the hypergeometric equation has a solution which is a polynomial of z whenever $-a$ or $-b$ is a non-negative integer. (If $a = -m$ or $b = -m$ and $c = -n$, where $n = 0, 1, 2, \ldots$, and $m = n+1, n+2, \ldots$, the series in (3) terminates.)

If a and b are different from $0, -1, -2, \ldots$, then the hypergeometric series (2) [or (3), in the case $c = -n$] converges absolutely for all values of $|z| < 1$. Since an application of 1.18(4) shows that

$$(5)\quad \frac{(a)_n (b)_n}{(c)_n \, n!} = \frac{\Gamma(a+n)}{\Gamma(a)} \frac{\Gamma(b+n)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(c+n)} \frac{\Gamma(1)}{\Gamma(n+1)} = \frac{\Gamma(c)\,\Gamma(1)}{\Gamma(a)\,\Gamma(b)} n^{a+b-c-1} \left[1 + O(n^{-1})\right],$$

we see by Raabe's test (see e.g. Bromwich 1947, pp. 39, 241) that for $|z| = 1$ we have:

absolute convergence for $|z| = 1$ if $\operatorname{Re}(a+b-c) < 0$,
conditional convergence for $|z| = 1$, $z \neq 1$ if $0 \leq \operatorname{Re}(a+b-c) < 1$,
divergence if $|z| = 1$ and $1 \leq \operatorname{Re}(a+b-c)$.

2.1.2. Elementary relations

From the definition (2) we have

$$F(a, b; c; z) = F(b, a; c; z).$$

The six functions

$$F(a \pm 1, b; c; z), \quad F(a, b \pm 1; c; z), \quad F(a, b; c \pm 1; z)$$

are called *contiguous* to $F(a, b; c; z)$. Between $F(a, b; c; z)$ and any two functions contiguous to it there exists a linear relation with coefficients which are linear functions of z. There are 15 relations of this type which have been found by Gauss. For a complete list see 2.8(31) to 2.8(45). One of these relations is

$$(6)\quad c\,F(a,b-1;c;z)+(a-b)\,z\,F(a,b;c+1;z)$$
$$=c\,F(a-1,b;c;z).$$

To verify (6) we expand both sides in a power series. Then the coefficient of z^n on the left-hand side of (6) is

$$c\,\frac{(a)_n\,(b-1)_n}{(c)_n\,n\,!}+(a-b)\,\frac{(a)_{n-1}\,(b)_{n-1}}{(c+1)_{n-1}\,(n-1)\,!}$$

$$=\frac{(a)_{n-1}\,(b)_{n-1}}{(c+1)_{n-1}\,(n-1)\,!}\,[a-b+(b-1)\,(a+n-1)/n]$$

$$=\frac{c\,(a)_{n-1}\,(b)_{n-1}}{(c)_n\,n\,!}\,(a-1)(b+n-1)=c\,\frac{(a-1)_n\,(b)_n}{(c)_n\,n\,!},$$

which proves (6).

If l, m, n, are integers, then

$$F(a+l,\ b+m;\ c+n;\ z)$$

can be expressed by repeated applications of these relations as a linear combination of $F(a,b;c;z)$ and one of its contiguous functions with coefficients which are rational functions of a, b, c, z.

Of course we must assume that c and $c+n$ are different from $0, -1, -2, \dots$. $F(a,b;c;z)$ and $F(a+l,\ b+m;\ c+n;\ z)$ are called *associated series*. It can be shown that any three associated series are connected by a linear homogeneous relation with polynomial coefficients provided that the values of the third parameter are different from $0, -1, -2, \dots$ (cf. Poole, 1936, p. 91 ff).

We also have

$$(7)\quad \frac{d^n}{dz^n}\,F(a,b;c;z)=(a)_n(b)_n\,[(c)_n]^{-1}\,F(a+n,\ b+n;c+n;z),$$

$$(8)\quad (a)_n\,z^{a-1}\,F(a+n,\ b;c;z)=\frac{d^n}{dz^n}\,[z^{a+n-1}\,F(a,b;c;z)],$$

$$(9)\quad (c)_n\,z^{c-1}\,(1-z)^{a+b-c}\,F(a,b;c;z)$$
$$=\frac{d^n}{dz^n}\,[z^{n+c-1}\,(1-z)^{n+a+b-c}\,F(a+n,\ b+n;c+n;z)].$$

Relation (9) is due to Jacobi (1859). For a complete list of such relations see 2.8(20) to 2.8(27). To prove (8) and (9) we introduce the operators

$$\delta = z\,\frac{d}{dz}, \quad D = \frac{d}{dz}.$$

We have

$$a\,F(a+1,b\,;\,c\,;\,z) = (\delta + a)\,F(a,b\,;\,c\,;\,z)$$

and since

$$(\delta + a)(\delta + a + 1)\cdots(\delta + a + n - 1)\,f(z) = z^{1-a}D^n[z^{a+n-1}f(z)]$$

for every analytic function $f(z)$ (cf. Poole, 1936, p. 93), this proves (8).

To obtain (9), we write (1) in the form

$$D\,[z(1-z)\,MD\,u] = abMu$$

where $M = z^{c-1}(1-z)^{a+b-c}$. According to (7), $D^{n-1}F(a,b\,;\,c\,;\,z)$ satisfies the hypergeometric equation with $a + n - 1$, $b + n - 1$, $c + n - 1$ instead of a, b, c and from that we obtain the recurrence relation

$$D[z^n(1-z)^n MD^n F] = (a+n-1)(b+n-1)\,[z(1-z)]^{n-1}MD^{n-1}F$$

and therefore

$$D^n[z^n(1-z)^n MD^n F] = (a)_n\,(b)_n\,MF.$$

Using (7) again and assuming that F is not a polynomial of degree less than n, i. e., $(a)_n\,(b)_n \neq 0$, we finally obtain (9).

The general theory of Riemann's equation (cf. section 2.6.1, and Poole, 1936) indicates that in general there must exist 24 solutions of (1) which are of the type

$$z^{\rho}(1-z)^{\sigma}F(a', b'; c';\ z')$$

where ρ, σ, a', b', c' are linear functions of a, b, c and where z and z' are connected by a homographic transformation. For a list of these 24 solutions (which are due to Kummer) see Goursat (1881), and 2.9(1) to 2.9(24). Any three of these solutions are connected by a linear relation with constant coefficients; for these see Goursat (1881) and 2.9(25) to 2.9(44). These relations can be used for the analytic continuation of the hypergeometric series, for a proof see 2.1.4.

2.1.3. The fundamental integral representations

If $\mathrm{Re}\,c > \mathrm{Re}\,b > 0$, we have Euler's formula

$$(10)\quad F(a,b\,;\,c\,;\,z) = \Gamma(c)\,[\Gamma(b)\,\Gamma(c-b)]^{-1}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt.$$

Here the right-hand side is a one-valued analytic function of z within the domain $|\arg\,(1-z)| < \pi$; therefore (10) gives also the analytic continuation of $F(a,b\,;\,c\,;\,z)$. To prove (10) for $|z| < 1$ we expand $(1-tz)^{-a}$ in a binomial series and integrate term by term; this leads to beta-integrals which can be evaluated by 1.5(1) to 1.5(5).

From the identity

$$(11)\quad \left\{z(1-z)\frac{\partial^2}{\partial z^2}+[c-(a+b+1)z]\frac{\partial}{\partial z}-ab\right\}$$

$$\times\,[t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}]=-a\frac{\partial}{\partial t}[t^{b}(1-t)^{c-b}(1-tz)^{-a-1}]$$

it follows that the right-hand side in (10) satisfies (1), and with $s=-t$ that

$$\int_0^\infty s^{b-1}(1+s)^{c-b-1}(1+sz)^{-a}\,ds$$

is a solution of (1) if Re $b>0$, Re$(a+1-c)>0$, and $|\arg z|<\pi$. With $s=\tau/(1-\tau)$ this becomes

$$\int_0^1 \tau^{b-1}(1-\tau)^{a-c}[1-\tau(1-z)]^{-a}\,dt,$$

and therefore

$$(12)\quad F(a,b;a+b+1-c;1-z)=\Gamma(a+b+1-c)\,[\Gamma(b)\,\Gamma(a+1-c)]^{-1}$$

$$\times\int_0^\infty s^{b-1}(1+s)^{c-b-1}(1+sz)^{-a}\,ds$$

is also a solution of the hypergeometric equation. Moreover, any integral

$$\int_C t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

is a solution of (1) if C is either closed on the Riemann surface of the integrand or terminates at zeros of $t^b(1-t)^{c-b}(1-tz)^{-a-1}$ Expanding $(1-tz)^{-a}$ in a binomial series and using the contour integrals 1.6(6) to 1.6(8) for the beta-function we find

$$F(a,b;c;z)=\frac{i\,\Gamma(c)\exp[i\pi(b-c)]}{\Gamma(b)\,\Gamma(c-b)\,2\sin\pi(c-b)}$$

$$\times\int_0^{(1+)} t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

$$\text{Re } b>0,\quad |\arg(1-z)|<\pi,\quad c-b\neq 1,2,3,\ldots,$$

$$F(a,b;c;z)=\frac{-i\,\Gamma(c)\exp(-i\pi b)}{\Gamma(b)\,\Gamma(c-b)\,2\sin\pi b}$$

$$\times\int_1^{(0+)} t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

$$\text{Re } c>\text{Re } b,\quad |\arg(-z)|<\pi,\quad b\neq 1,2,3,\ldots,$$

$$(13)\quad F(a,b;c;z)=\frac{-\Gamma(c)\exp(-i\pi c)}{\Gamma(b)\,\Gamma(c-b)\,4\sin\pi b\,\sin\pi(c-b)}$$

$$\times\int^{(1+,0+,1-,0-)} t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

$$|\arg(-z)|<\pi,\quad b,\ 1-c,\ c-b\neq 1,2,3,\ldots.$$